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Formal Hopf algebra theory, II: Lax centres

Ignacio L. López Franco

Laboratorie Prueves, Programmes et Systèmes, Université Paris Diderot – Paris 7, Case 7014, 75205 Paris Cedex 13, France

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ABSTRACT

This paper is the second in a series started by [Ignacio L. López Franco, Formal Hopf algebra theory I: Hopf modules for pseudomonoids, J. Pure Appl. Algebra 213 (2009) 1046–1063], aiming to extend the basic theory of Hopf algebras to the context of pseudomonoids in monoidal bicategories. This article concentrates on the notion of lax centre of a pseudomonoid and its relationship with the Drinfel'd or quantum double of a finite Hopf algebra and the centre of a monoidal category. We can distinguish two parts in the present paper. In the first, for a pseudomonoid A with lax centre $Z_\ell A$ in a Gray monoid $\mathscr{M}$ with certain extra properties, we exhibit an equivalence $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$ of categories enriched in $\mathscr{M}(I, I)$. In the second, we construct the lax centre of a left autonomous map pseudomonoid A as an Eilenberg–Moore object for a certain opmonoidal monad on A. Moreover, if A is also right autonomous, the lax centre coincides with the centre. As an application, we show that a (left) autonomous monoidal $\mathscr{V}$-category has a (lax) centre in $\mathscr{V}$-**Mod**, of which we give an explicit description. In another application, we prove that a finite-dimensional coquasi-Hopf algebra H has a centre in the monoidal bicategory **Comod**(**Vect**) and it is equivalent to the Drinfel'd double $D(H)$.

1. Introduction

This paper is the second in a series aimed to extend the basic theory of Hopf algebras to the context of pseudomonoids in monoidal bicategories. We exploit the fact that autonomous pseudomonoids are generalised Hopf algebras to extend some of the classical constructions of Hopf algebra theory to the context of pseudomonoids. We use the results in [13] to study centres and lax centres of autonomous map pseudomonoids, and their relationship with the Drinfel'd double.

A classical notion of the centre of an algebraic structure is the centre of a monoid. If M is a monoid, its centre is the set of elements of M with the *property* of commuting with every element of M. We can slightly change our point of view and said that the centre of M is the set whose elements are pairs $(x, (x \cdot -) = (- \cdot x))$: elements of $x \in M$ equipped with the extra *structure* of an equality between the multiplication with x on the left and on the right. The centre of a monoidal category, defined in [11], follows the spirit of the latter point of view: from the algebraic structure of a monoidal category $\mathscr{C}$ one forms a new algebraic structure $Z\mathscr{C}$, called the centre of $\mathscr{C}$. We have a functor $Z\mathscr{C} \to \mathscr{C}$, and $Z\mathscr{C}$ has a monoidal structure such that this functor is strong monoidal. Moreover, $Z\mathscr{C}$ has a canonical braiding. The objects of $Z\mathscr{C}$ are pairs (x, γ_x) where $\gamma_x : (- \otimes x) \Rightarrow (x \otimes -)$ is an invertible natural transformation. In this context one can also consider the *lax centre* $Z_\ell\mathscr{C}$ of $\mathscr{C}$, simply by dropping the requirement of the invertibility of γ_x. See Example 3.3. The functor $Z\mathscr{C} \to \mathscr{C}$ is the universal one satisfying certain commutation properties.

Another classically considered centre-like object is the Drinfel'd double of a finite-dimensional Hopf algebra, or, more recently, of a (co)quasi-Hopf algebra. See [15,18]. Here the concept is not the one of the object classifying morphisms with certain commutation properties, but it is a representational one. Roughly speaking, the Drinfel'd double of a finite-dimensional Hopf algebra H is a Hopf algebra $D(H)$ such that the category of representations of $D(H)$ is monoidally equivalent to the centre of the category of representations of H.

E-mail address: I.Lopez-Franco@dpmms.cam.ac.uk.

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We study lax centres $Z_\ell A$ of a map pseudomonoid A in a braided Gray monoid $\mathscr{M}$ from two points of view. Firstly we would like to have canonical equivalences $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$. The simple minded choice is to take the object on the right-hand side of the equivalence as the lax centre of the monoidal category $\mathscr{M}(I, A)$. However, this turns out to be insufficient to obtain an equivalence. We are led to consider $\mathscr{M}(I, A)$ as an $\mathscr{M}(I, I)$-enriched category and its lax centre in $\mathscr{M}(I, I)$-**Cat**. This context provides an enriched equivalence as above, at the price of requiring certain mild conditions on $\mathscr{M}$. We apply these constructions to (pro)monoidal enriched categories.

Secondly, we construct lax centres of autonomous map pseudomonoids. By means of the Hopf module construction of [13], we construct the lax centre as an internal analogue of the category of two sided Hopf modules. This generalises the fact that for a finite Hopf algebra the category of two sided Hopf modules is monoidally equivalent to the centre of the category of representations of the Hopf algebra (and to the category of representations of the Drinfel'd double of the Hopf algebra). We show that the (lax) centre of a finite-dimensional coquasi-Hopf algebra H always exists within the bicategory of comodules. Moreover, the construction of this centre is explicit, can be taking to be finite-dimensional, and it is isomorphic as a coalgebra and equivalent as a coquasibialgebra to the Drinfel'd double of H.

Now we describe the organisation of the paper.

In Section 2 we give the minimal necessary background in Gray monoids and pseudomonoids. In Section 3 we introduce lax centres of pseudomonoids and give the first examples.

Section 4 studies the relationship between $\mathscr{M}(I, Z_\ell A)$ and the centre of the monoidal category $\mathscr{M}(I, A)$. We show that the universal $Z_\ell A \to A$ induces an equivalence between the categories above, when we consider them as $\mathscr{M}(I, I)$-enriched categories.

Section 5 recalls some of the results in [13]. Using these results, Section 6 exhibits lax centres of left autonomous map pseudomonoids as Eilenberg–Moore constructions for a certain monad. When the pseudomonoid is also right autonomous, the lax centre coincides with the centre.

The last two sections are dedicated to examples. Section 7 deals with the example of the bicategory $\mathscr{V}$-**Mod** of $\mathscr{V}$-modules. We show that if the lax centre $Z_\ell\mathscr{A}$ (in $\mathscr{V}$-**Mod**) of a promonoidal $\mathscr{V}$-category $\mathscr{A}$ exists, then there is an equivalence of $\mathscr{V}$-categories $[Z_\ell\mathscr{A}, \mathscr{V}] \simeq Z_\ell[\mathscr{A}, \mathscr{V}]$; here the $\mathscr{V}$-category on the right-hand side is the lax centre in $\mathscr{V}$-**Cat**. We also prove that a left autonomous monoidal $\mathscr{V}$-category always has a lax centre in $\mathscr{V}$-**Mod**. An explicit description of this lax centre is given.

Section 8 studies the example of the bicategory of comodules **Comod**($\mathscr{V}$), and the relationship between centres and Drinfel'd doubles. The main result is that if H is a finite-dimensional coquasi-Hopf algebra, then its centre in **Comod**($\mathscr{V}$) exists and it is isomorphic as coalgebra and equivalent as coquasibialgebra to $D(H)$, the Drinfel'd double of H.

2. Background on Gray monoids and pseudomonoids

In this section we recall the basic constructions we use along the article.

2.1. Gray monoids

A Gray monoid, sometimes called a semi-strict monoidal bicategory, is a 2-category $\mathscr{M}$ equipped with a tensor product pseudofunctor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ and a unit object I satisfying axioms. Gray monoids are formally defined as monoids in the monoidal category **Gray** of [9], or equivalently as a **Gray**-enriched category with one object. An elementary definition was provided in [6]. Gray monoids play a central role in the theory of monoidal bicategories in that every monoidal bicategory (as defined in [9]) is monoidally biequivalent to a Gray monoid. This is a particular instance of the main result of [9].

A braided Gray monoid is a Gray monoid equipped with pseudonatural equivalences $c_{X,Y} : X \otimes Y \to Y \otimes X$ and invertible 2-cells

$$(X \otimes c_{W\otimes Y,Z})(c_{W,X} \otimes Y \otimes Z) \cong (c_{W,X\otimes Z} \otimes Y)(W \otimes X \otimes c_{Y,Z})$$

satisfying three axioms. These axioms imply that the tensor product pseudofunctor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is (strong) monoidal, with monoidal constraints $X \otimes c_{Y,Z} \otimes W : X \otimes Y \otimes Z \otimes W \to X \otimes Z \otimes Y \otimes W$ and $1 : I \otimes I \to I$.

We shall also use the notion of closed Gray monoid as defined in [6]. A Gray monoid is right closed when each 2-functor $X \otimes -$ has a right biadjoint $[X, -]$. When this right biadjoint is of the form $Y \otimes -$ we say that Y is a right bidual of X, and we denote it by X°. We denote the resulting evaluation and coevaluation 1-cells by $\mathrm{e} : X \otimes X^\circ \to I$ and $\mathrm{n} : I \to X^\circ \otimes X$ respectively.

2.2. Pseudomonoids and dualizations

The fundamental structures we will consider in a Gray monoid are *pseudomonoids*. The notion of pseudomonoid was defined in [6], and consists of an object A in a Gray monoid with a multiplication $p : A \otimes A \to A$ and a unit $j : I \to A$, and invertible 2-cells $\phi : p(p \otimes A) \cong p(A \otimes p)$, $p(j \otimes A) \cong 1_A \cong p(A \otimes j)$ satisfying two axioms. The canonical example of a pseudomonoid is a monoidal category in **Cat**, where the latter is considered as a monoidal bicategory via the cartesian product. Two further examples are promonoidal (enriched) categories [3] and coquasibialgebras, which are pseudomonoids in the monoidal bicategories of $\mathscr{V}$-modules $\mathscr{V}$-**Mod** and of comonoids **Comon**($\mathscr{V}$) respectively.

In the same way as monoidal $\mathscr{V}$-categories are pseudomonoids in $\mathscr{V}$-**Cat** and hence $\mathscr{V}$-**Mod**, left autonomous monoidal $\mathscr{V}$-categories correspond to the notion of a left autonomous pseudomonoid, introduced in [4]. If A is a pseudomonoid with right bidual, a left dualization for A is a 1-cell $d : A^{\circ} \to A$ with 2-cells $\alpha : p(d \otimes A)\mathrm{n} \Rightarrow j$, $\beta : j\mathrm{e} \Rightarrow p(A \otimes d)$ satisfying two axioms. These axioms can be better understood using the extraordinary 2-cells of [21]. If we write $f \bullet g$ for the composite $p(f \otimes A)(X \otimes g) : X \otimes Y \to A$, for a pair of arrows $f : X \to A$, $g : Y \to A$, the 2-cells α, β are extraordinary 2-cells $\alpha : d \bullet 1_A \to j$ and $\beta : j \to A \bullet d$. The axioms of a left dualization state that α, β satisfy the usual triangular equalities of an adjunction, expressed as the composite of ordinary and extraordinary 2-cells below

$$1 = \left(1_A \xrightarrow{\cong} j \bullet 1_A \xrightarrow{\beta \bullet 1_A} (1_A \bullet d) \bullet 1_A \xrightarrow{\cong} 1_A \bullet (d \bullet 1_A) \xrightarrow{1_A \bullet \alpha} 1_A \bullet j \xrightarrow{1_A} 1_A\right)$$

$$1 = \left(d \xrightarrow{\cong} d \bullet j \xrightarrow{d \bullet \beta} d \bullet (1_A \bullet d) \xrightarrow{\cong} (d \bullet 1_A) \bullet d \xrightarrow{\alpha \bullet d} j \bullet d \xrightarrow{\cong} d\right).$$

A left autonomous pseudomonoid is a pseudomonoid equipped with a left dualization. As observed in [4], if a dualization exists, then it is unique up to canonical isomorphisms. Examples of a left autonomous pseudomonoid, that underlie the importance of the notion, are the (coquasi-)Hopf algebras.

3. Centres and lax centres

We shall work in a *braided Gray monoid*, in the sense of [6]. See Section 2.1. The centre of a pseudomonoid was defined in [22]. Here we will be interested in the lax version of the centre, called the *lax centre* of a pseudomonoid. The definition is exactly the same as that of the centre but for the fact that we drop the requirement of the invertibility of certain 2-cells.

Definition 3.1. Given a pseudomonoid in a braided Gray monoid $\mathscr{M}$ we define for each object X a category $CP_\ell(X, A)$. The objects, called *lax centre pieces*, are pairs (f, γ) where $f : X \to A$ is a 1-cell and γ is a 2-cell

$$\begin{tikzcd}
A \otimes X \arrow[d, "1\otimes f"'] & & X \otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes 1"] \\
A \otimes A \arrow[dr, "p"'] \arrow[rr, phantom, "\Leftarrow_{\gamma}"] & & A \otimes A \arrow[dl, "p"] \\
& A &
\end{tikzcd} \tag{1}$$

satisfying axioms (2) and (3) in Fig. 1. The arrows $(f, \gamma) \to (f', \gamma')$ are the 2-cells $f \Rightarrow f'$ which are compatible with γ and γ' in the obvious sense.

This is the object part of a pseudofunctor $CP_\ell(-, A) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$, that is defined on 1-cells and 2-cells just by precomposition. When CP_ℓ is birepresentable we call a birepresentation $z_\ell : Z_\ell A \to A$ a *lax centre* of the pseudomonoid A.

A *centre piece* is a lax centre piece (f, γ) such that γ is invertible. The full subcategories $CP(X, A) \subset CP_\ell(X, A)$ with objects the centre pieces define a pseudofunctor $CP(-, A) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$, and we call a birepresentation of it a *centre* of A, denoted by $z : ZA \to A$.

Definition 3.2. The inclusion $CP(-, A) \hookrightarrow CP_\ell(-, A)$ induces a 1-cell $z_c : ZA \to Z_\ell A$, unique up to isomorphism, such that $z_\ell z_c \cong z$ as lax centre pieces. When z_c is an equivalence we will say that the centre of A coincides with the lax centre.

Example 3.3. The centre of a pseudomonoid in **Cat**, that is, of a monoidal category, is the usual centre defined in [11]. In fact, lax centres and centres of pseudomonoids in $\mathscr{V}$-**Cat** exist and are given by the constructions in [5]. Lax centres or (ordinary) monoidal categories were also considered in [18] under the name of 'weak centers'. If $\mathscr{A}$ is a monoidal $\mathscr{V}$-category, its lax centre $Z_\ell\mathscr{C}$ has as objects the pairs (x, γ) where x is an object of $\mathscr{C}$ and $\gamma : (-\otimes x) \Rightarrow (x \otimes -)$ is a $\mathscr{V}$-natural transformation. The $\mathscr{V}$-enriched hom $Z_\ell\mathscr{C}((x, \gamma), (y, \delta))$ is the equalizer of the pair of arrows

$$\begin{tikzcd}
\mathscr{C}(x,y) \arrow[r] \arrow[d] & {[\mathscr{C},\mathscr{C}](-\otimes x, -\otimes y)} \arrow[d, "{[\mathscr{C},\mathscr{C}](\gamma,1)}"] \\
{[\mathscr{C},\mathscr{C}](x\otimes -, y\otimes -)} \arrow[r, "{[\mathscr{C},\mathscr{C}](1,\delta)}"'] & {[\mathscr{C},\mathscr{C}](x\otimes -, -\otimes y)}
\end{tikzcd}$$

Observation 3.4. By [22], in a monoidal closed bicategory with finite limits, *every* pseudomonoid has a centre.

4. Lax centres of convolution monoidal categories

For any pseudomonoid (A, j, p) in a Gray monoid $\mathscr{M}$ we know from [6] that the category $\mathscr{M}(I, A)$ has a canonical *convolution* monoidal structure. The tensor product is given by $f * g = p(f \otimes A)g$ with unit j. We would like to exhibit an equivalence $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$. Our leading example is the bicategory $\mathscr{V}$-**Mod** of $\mathscr{V}$-categories and $\mathscr{V}$-modules. In this example the tensor product just described is just Day's convolution tensor product introduced in [3]. For details about this bicategory see Section 7. Henceforth, we shall assume our Gray monoid $\mathscr{M}$ satisfies additional properties, which we explain below.

$$
\begin{tikzcd}
A\otimes A\otimes X \arrow[d, "1\otimes 1\otimes f"'] \arrow[dr, "p\otimes 1"] & & & & X\otimes A\otimes A \arrow[llll, "c_{X,A\otimes A}"'] \arrow[dl, "1\otimes p"'] \arrow[d, "f\otimes 1\otimes 1"] \\
A\otimes A\otimes A \arrow[r, phantom, "\cong"] \arrow[d, "1\otimes p"'] \arrow[dr, "p\otimes 1"'] & A\otimes X \arrow[d, "1\otimes f"] & \cong & X\otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes 1"'] & A\otimes A\otimes A \arrow[l, phantom, "\cong"] \arrow[dl, "1\otimes p"] \arrow[d, "p\otimes 1"] \\
A\otimes A \arrow[r, phantom, "\cong"] \arrow[drr, "p"'] & A\otimes A \arrow[dr, "p"] \arrow[rr, phantom, "\Leftarrow_{\gamma}"] & & A\otimes A \arrow[dl, "p"'] & A\otimes A \arrow[l, phantom, "\cong"] \arrow[dll, "p"] \\
& & A & &
\end{tikzcd}
$$

$$\Vert \tag{2}$$

$$
\begin{tikzcd}
A\otimes A\otimes X \arrow[d, "1\otimes 1\otimes f"'] & & A\otimes X\otimes A \arrow[ll, "1\otimes c_{X,A}"'] \arrow[d, "1\otimes f\otimes 1"] & & X\otimes A\otimes A \arrow[ll, "c_{X,A}\otimes 1"'] \arrow[d, "f\otimes 1\otimes 1"] \arrow[llll, bend right, "c_{X,A\otimes A}"', "\cong"] \\
A\otimes A\otimes A \arrow[dr, "1\otimes p"'] \arrow[rr, phantom, "\Leftarrow_{1\otimes\gamma}"] & & A\otimes A\otimes A \arrow[dl, "1\otimes p"'] \arrow[dr, "p\otimes 1"] \arrow[rr, phantom, "\Leftarrow_{\gamma\otimes 1}"] & & A\otimes A\otimes A \arrow[dl, "p\otimes 1"] \\
& A\otimes A \arrow[dr, "p"'] \arrow[rr, phantom, "\cong"] & & A\otimes A \arrow[dl, "p"] & \\
& & A & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[rrrr, equal] \arrow[dr, "j\otimes 1"] \arrow[dd, "f"'] & & \cong & & X \arrow[dl, "1\otimes j"'] \arrow[dd, "f"] \\
\cong & A\otimes X \arrow[d, "1\otimes f"'] & & X\otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes 1"] & \cong \\
A \arrow[r, "j\otimes 1"] \arrow[drr, bend right, "1"', "\cong"] & A\otimes A \arrow[dr, "p"] \arrow[rr, phantom, "\Leftarrow_{\gamma}"] & & A\otimes A \arrow[dl, "p"'] & A \arrow[l, "1\otimes j"'] \arrow[dll, bend left, "1", "\cong"'] \\
& & A & &
\end{tikzcd}
= 1_f \tag{3}
$$

Fig. 1. Lax centre piece axioms.

Recall that a 2-cell

$$
\begin{tikzcd}
& Y \arrow[dl, "{}^{f}g"'] \arrow[d, "g"] \arrow[dl, phantom, shift left=3, "\Downarrow\lambda"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

in a bicategory $\mathscr{B}$ is said to *exhibit ${}^{f}g$ as the right lifting of g through f* if it induces a bijection $\mathscr{B}(Y,X)(k,{}^{f}g) \cong \mathscr{B}(Y,Z)(fk,g)$, natural in k. Clearly, right liftings are unique up to compatible isomorphisms. See [24].

We shall assume that our braided Gray monoid $\mathscr{M}$ is closed (see Section 2.1 and the references therein) and has *right liftings* of arrows out of I through arrows out of I. As explained in [6], this endows each $\mathscr{M}(X,Y)$ with the structure of a $\mathscr{V}$-category where $\mathscr{V} = \mathscr{M}(I,I)$ is a symmetric monoidal closed category whose tensor product is given by composition. We denote this $\mathscr{V}$-category by $\mathbb{M}(X,Y)$. The $\mathscr{V}$-enriched hom $\mathbb{M}(X,Y)(f,g)$ is ${}^{\hat{f}}\hat{g}$, the right lifting of $\hat{g} : I \to [X,Y]$ through $\hat{f} : I \to [X,Y]$, where these two arrows correspond to f and g under the closedness biadjunction. Both $\hat{f}$ and $\hat{g}$ are determined up to isomorphism, and then so is $\mathbb{M}(X,Y)(f,g)$. The compositions $\mathbb{M}(X,Y)(g,h) \otimes \mathbb{M}(X,Y)(f,g) \to \mathbb{M}(X,Y)(f,h)$ and units $1_I \to \mathbb{M}(X,Y)(f,f)$, along with the $\mathscr{V}$-category axioms, are easily deduced from the universal property of the right liftings. Observe that the underlying category of the $\mathscr{V}$-category $\mathbb{M}(X,Y)$ is the hom-category $\mathscr{M}(X,Y)$. For, $\mathscr{V}(1_I, \mathbb{M}(X,Y)(f,g)) = \mathscr{V}(1_I, {}^{\hat{f}}\hat{g}) \cong \mathscr{M}(I,[X,Y])(\hat{f},\hat{g}) \cong \mathscr{M}(X,Y)(f,g)$.

One can define *composition $\mathscr{V}$-functors* $\mathbb{M}(Y,Z) \otimes \mathbb{M}(X,Y) \to \mathbb{M}(X,Z)$ on objects just by composition in $\mathscr{M}$ and on $\mathscr{V}$-enriched homs in the following way. Given $f,h : Y \to Z$ and $g,k : X \to Y$, define an arrow $\mathbb{M}(I,[Y,Z])(\hat{f},\hat{h}) \otimes$

$\mathbb{M}(I,[X,Y])(\hat{g},\hat{k}) \to \mathbb{M}(I,[X,Z])(\widehat{fg},\widehat{hk})$ as the 2-cell in $\mathscr{M}$ corresponding to the following pasting.

$$
\begin{tikzcd}
 & & I \arrow[dl, "{}^{g}\hat{k}"'] \arrow[d, "\hat{k}"] \arrow[ddr, "\widehat{hk}"] \arrow[dl, phantom, shift right=3ex, "\Downarrow"] & \\
 & I \arrow[r, "\hat{g}"'] \arrow[dl, "{}^{f}\hat{h}"'] \arrow[d, "\hat{h}"] \arrow[dl, phantom, shift left=3ex, "\Downarrow"] \arrow[dr, phantom, "\cong"] & {[X,Y]} \arrow[d, "\hat{h}\otimes 1"] \arrow[dr, phantom, "\cong"] & \\
I \arrow[r, "\hat{f}"] \arrow[rrr, bend right, "\widehat{fg}"'] & {[Y,Z]} \arrow[r, "1\otimes\hat{g}"] & {[Y,Z]\otimes[X,Y]} \arrow[r, "\mathrm{comp}"] \arrow[d, phantom, "\cong"] & {[X,Z]} \\
 & & &
\end{tikzcd}
$$

There are also *identity $\mathscr{V}$-functors* from the trivial $\mathscr{V}$-category to $\mathbb{M}(X,X)$. On objects they just pick the identity 1-cells 1_X and on homs they are given by the arrows $1_I \to {}^{(\hat{1_X})}\hat{1_X}$ corresponding to the identity 2-cells $\hat{1_X} \Rightarrow \hat{1_X}$. These composition and identity $\mathscr{V}$-functors endow $\mathscr{M}$ with the structure of a category $\mathbb{M}$ weakly enriched in $\mathscr{V}$-**Cat** in the sense that the category axioms hold only up to specified coherent $\mathscr{V}$-natural isomorphisms. For example, when $\mathscr{V}$ is the category of sets, we get a (locally small) bicategory.

Now we shall further suppose that the category $\mathscr{V} = \mathscr{M}(I,I)$ is complete. This allows us to consider functor $\mathscr{V}$-categories. In this situation, the composition $\mathscr{V}$-functors induce $\mathscr{V}$-functors $\mathbb{M}(X,-)_{Y,Z} : \mathbb{M}(Y,Z) \to [\mathbb{M}(X,Y),\mathbb{M}(X,Z)]$ making the pseudofunctor $\mathscr{M}(X,-) : \mathscr{M} \to \mathscr{V}$-**Cat** locally a $\mathscr{V}$-functor.

Lemma 4.1. *Under the hypotheses above, if A is a pseudomonoid in $\mathscr{M}$, $CP_\ell(I,A)$ has a canonical structure of a $\mathscr{V}$-category such that the forgetful functor $CP_\ell(I,A) \to \mathscr{M}(I,A)$ is the underlying functor of a $\mathscr{V}$-functor. Moreover, $CP(I,A)$ is a full sub-$\mathscr{V}$-category of $CP_\ell(I,A)$.*

Proof. We give only a sketch of a proof; the details are an exercise in the universal property of right liftings. Given two lax centre pieces (f,α) and (g,β), define the $\mathscr{V}$-enriched hom $CP_\ell(I,A)((f,\alpha),(g,\beta))$ as the equalizer in $\mathscr{V}$ of the pair

$$
\begin{tikzcd}
\mathbb{M}(I,A)(f,g) \arrow[r] \arrow[d] & \mathbb{M}(A,A)(p(A\otimes f),p(A\otimes g)) \arrow[d, "\mathbb{M}(A,A)(\alpha,1)"] \\
\mathbb{M}(A,A)(p(f\otimes A),p(g\otimes A)) \arrow[r, "\mathbb{M}(A,A)(1,\beta)"'] & \mathbb{M}(A,A)(p(f\otimes A),p(A\otimes g))
\end{tikzcd}
\tag{4}
$$

where the unlabelled arrows are induced by the universal property of right liftings under postcomposition with the arrows $A \to [A,A]$ corresponding to p and $pc_{A,A}$. With this definition, an arrow $1_I \to CP_\ell(I,A)((f,\alpha),(g,\beta))$ in $\mathscr{V} = \mathscr{M}(I,I)$ corresponds to an arrow $(f,\alpha) \to (g,\beta)$ in the ordinary category $CP_\ell(I,A)$. The composite $CP_\ell(I,A)((g,\beta),(h,\gamma)) \otimes CP_\ell(I,A)((f,\alpha),(g,\beta)) \to CP_\ell(I,A)((f,\alpha),(h,\gamma))$ is induced by the composition $\mathscr{M}(I,A)(g,h) \otimes \mathscr{M}(I,A)(f,g) \to \mathscr{M}(I,A)(f,h)$ and the universal property of the equalizers, and likewise for the identities. □

Proposition 4.2. *Assume the lax centre of A exists, with universal centre piece (z_ℓ,γ). Under the hypothesis above, (z_ℓ,γ) induces a $\mathscr{V}$-enriched equivalence U making the following diagram commute.*

$$
\begin{tikzcd}
\mathbb{M}(I,Z_\ell A) \arrow[rr, "U"] \arrow[dr, "\mathbb{M}(I,z_\ell)"'] & & CP_\ell(I,A) \arrow[dl] \\
 & \mathbb{M}(I,A) &
\end{tikzcd}
$$

Moreover, the same holds if the centre of A exists and we use ZA and $CP(I,A)$ instead of $Z_\ell A$ and $CP_\ell(I,A)$.

Proof. On objects, U is equal to the usual functor, that is, it sends $f : I \to Z_\ell A$ to the lax centre piece $(z_\ell f, \gamma(f\otimes A))$. Next we describe U on $\mathscr{V}$-enriched homs. Define ϱ by the following equality, where π exhibits ${}^hk$ as a right lifting of k through h and ϖ exhibits ${}^{(z_\ell h)}(z_\ell k)$ as a right lifting of $z_\ell k$ through $z_\ell h$.

$$
\begin{tikzcd}
 & I \arrow[dl, "{}^hk"'] \arrow[d, "k"] \arrow[dl, phantom, shift left=3ex, "\Downarrow\pi"] & \\
I \arrow[r, "h"'] & Z_\ell A \arrow[dr, "z_\ell"'] & \\
 & & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & I \arrow[ddll, bend right, "{}^hk"'] \arrow[ddll, "{}^{(z_\ell h)}(z_\ell k)"] \arrow[d, "k"] \arrow[ddll, phantom, bend right=20, "\Downarrow\varrho"] \\
 & & Z_\ell A \arrow[d, "z_\ell"] \arrow[dl, phantom, "\Downarrow\varpi"] \\
I \arrow[r, "h"'] & Z_\ell A \arrow[r, "z_\ell"'] & A
\end{tikzcd}
\tag{5}
$$

This pasted composite is trivially a morphism of lax centre pieces $U(h({}^hk)) \to U(k)$, and this means exactly that ϱ factors through the equalizer

$$CP_\ell(I,A)(U(h),U(k)) \rightarrowtail {}^{(z_\ell h)}(z_\ell k) = \mathbb{M}(I,A)(z_\ell h, z_\ell k)$$

$$\begin{tikzcd}
 & & A \arrow[ddll, bend right=40, "v\otimes 1"'] \arrow[ddll, "(z_\ell h)(z_\ell k)\otimes 1"] \arrow[d, "k\otimes 1"] \arrow[drr, "1\otimes k"] \arrow[dll, phantom, "\Downarrow\nu\otimes 1"] \arrow[dr, phantom, "\simeq"] & & \\
 & \arrow[r, phantom, "\Downarrow\varpi\otimes 1"] & Z_\ell A\otimes A \arrow[rr, "\simeq"] \arrow[d, "z_\ell\otimes 1"] & & A\otimes Z_\ell A \arrow[d, "1\otimes z_\ell"] \\
A \arrow[r, "h\otimes 1"'] & Z_\ell A\otimes A \arrow[r, "z_\ell\otimes 1"'] & A^2 \arrow[dr, "p"'] \arrow[rr, phantom, "\overset{\gamma}{\Rightarrow}"] & & A^2 \arrow[dl, "p"] \\
 & & & A &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & & A \arrow[ddll, bend right=40, "1\otimes v"'] \arrow[ddll, "1\otimes {}^{(z_\ell h)}(z_\ell k)"] \arrow[d, "1\otimes k"] \arrow[dl, phantom, "\Downarrow 1\otimes\nu"] & \\
 & \arrow[r, phantom, "\Downarrow 1\otimes\varpi"] & A\otimes Z_\ell A \arrow[d, "1\otimes z_\ell"] & \\
A \arrow[r, "1\otimes h"] \arrow[dr, "h\otimes 1"'] & A\otimes Z_\ell A \arrow[r, "1\otimes z_\ell"] & A^2 \arrow[r, "p"] & A \\
 & Z_\ell A\otimes A \arrow[u, "\simeq"] \arrow[r, "z_\ell\otimes 1"'] & A^2 \arrow[ur, "p"'] \arrow[u, phantom, "\gamma\Uparrow"] &
\end{tikzcd}$$

Fig. 2.

in (4). Denote by $\tilde{\varrho} : {}^{h}k = \mathscr{M}(I, A)(h, k) \to CP_\ell(I, A)(U(h), U(k))$ the resulting arrow in $\mathscr{V}$. This is by definition the effect of U on enriched homs.

Observe that the underlying ordinary functor of U is the usual equivalence given by the universal property of the lax centre. Hence, U is essentially surjective on objects as a $\mathscr{V}$-functor. It is sufficient, then, to show that U is fully faithful, or, in other words, that $\tilde{\varrho}$ is invertible. To do this, we shall show that ϱ has the universal property of the equalizer defining $CP_\ell(I, A)(U(h), U(k))$.

Suppose $\nu : v \to {}^{(z_\ell h)}(z_\ell k)$ is an arrow in $\mathscr{V}$ equalizing the pair of arrows ${}^{(z_\ell h)}(z_\ell k) \to \mathbb{M}(A, A)(p(z_\ell h \otimes A), p(A \otimes z_\ell k))$ analogous to (4). If one unravels this condition, one gets the equality in Fig. 2. This means that the 2-cell $\varpi \cdot (z_\ell h\nu)$ is an arrow in the ordinary category $CP_\ell(I, A)$ from $U(hv) = (z_\ell hv, \gamma((hv) \otimes A))$ to $U(k) = (z_\ell k, \gamma(k \otimes A))$, and therefore there exists a unique 2-cell $\tau : hv \Rightarrow k : I \to Z_\ell A$ such that $z_\ell\tau = \varpi \cdot (z_\ell h\nu)$. From the universal property of right liftings, we deduce the existence of a unique $\tau' : v \Rightarrow {}^{h}k$ such that $\pi \cdot (h\tau') = \tau$. In order to show that $\varrho : {}^{h}k \Rightarrow {}^{(z_\ell h)}(z_\ell k)$ has the universal property of the equalizer as explained above, we have to show that $\varrho\tau' = \nu$. But the pasting of $\varrho \cdot \tau'$ with ϖ, $\varpi \cdot (z_\ell h(\varrho \cdot \tau'))$, is equal, by definition of ϱ, to $z_\ell(\pi \cdot (h\tau')) = z_\ell\tau = \varpi \cdot (z_\ell h\nu)$. It follows that $\varrho \cdot \tau' = \nu$.

The case of the centre is completely analogous to that of the lax centre. The $\mathscr{V}$-functor U is defined on objects by sending $f : I \to ZA$ to the centre piece $(zf, \gamma(f \otimes A))$, where (z, γ) is the universal centre piece. The definition of U on $\mathscr{V}$-enriched homs is the same as in the case of the lax centre above. $\square$

In order to exhibit the desired equivalence $\mathbb{M}(I, Z_\ell A) \simeq Z_\ell(\mathbb{M}(I, A))$, we shall require of our closed braided Gray monoid $\mathscr{M}$ two further properties.

Condition 1. Firstly, the pseudofunctor $\mathscr{M}(I, -) : \mathscr{M} \to \mathscr{V}$-**Cat** must be locally faithful. In other words, for every pair of 1-cells f, g, the following must be a monic arrow in $\mathscr{V}$:

$$\mathbb{M}(X, Y)(f, g) \to [\mathbb{M}(I, X), \mathbb{M}(I, Y)](\mathbb{M}(I, f), \mathbb{M}(I, g)). \tag{6}$$

Condition 2. Secondly, for any $f, g : X \to Y$, the image of the arrow (6) under $\mathscr{V}(I, -) : \mathscr{V} \to$ **Set** must be surjective. This condition is saying that every $\mathscr{V}$-natural transformation $\mathscr{M}(I, f) \Rightarrow \mathscr{M}(I, g)$ is induced by a 2-cell $f \Rightarrow g$; this 2-cell is unique by Condition 1.

All these properties are satisfied by our main example of $\mathscr{V}$-**Mod**, as we shall see later.

Theorem 4.3. *Under the hypothesis above, if A has a lax centre then there exists a $\mathscr{V}$-enriched equivalence making the following diagram commute up to a canonical isomorphism.*

$$\begin{tikzcd}
\mathbb{M}(I, Z_\ell A) \arrow[rr, "\simeq"] \arrow[dr, "\mathbb{M}(I,z_\ell)"'] & & Z_\ell(\mathbb{M}(I, A)) \arrow[dl, "V"] \\
 & \mathbb{M}(I, A) &
\end{tikzcd}$$

*Here the $\mathscr{V}$-category on the right-hand side is a lax centre in $\mathscr{V}$-**Cat** and V is the forgetful $\mathscr{V}$-functor. Furthermore, the result remains true if we write centres in place of lax centres.*

Proof. By Proposition 4.2 it is enough to exhibit a $\mathscr{V}$-enriched equivalence between $CP_\ell(I, A)$ and $Z_\ell(\mathbb{M}(I, A))$ commuting with the forgetful functors.

Define a $\mathscr{V}$-functor $\Phi : CP_\ell(I, A) \to Z_\ell(\mathbb{M}(I, A))$ as follows. On objects $\Phi(f, \alpha) = (f, \Phi_1(\alpha))$ where

$$\Phi_1(\alpha)_h : h * f \cong p(A \otimes f)h \xrightarrow{\alpha h} p(f \otimes A)h \cong f * h.$$

Recall that the $\mathscr{V}$-enriched hom $CP_\ell(I, A)((f, \alpha), (g, \beta))$ is the equalizer of (4) and $Z_\ell(\mathbb{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta))$ is the equalizer of the diagram in Example 3.3, where $\mathscr{C} = \mathbb{M}(I, A)$, $x = f$, $y = g$, $\gamma = \Phi_1(\alpha)$ and $\delta = \Phi_1(\beta)$. We can draw a diagram

$$\begin{tikzcd}
\mathbb{M}(I,A)(f,g) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \mathbb{M}(A,A)(p(f\otimes A), p(A\otimes g)) \arrow[d, "\mathbb{M}(I,-)"] \\
 & {[\mathbb{M}(I,A), \mathbb{M}(I,A)](f * -, - * g)}
\end{tikzcd}$$

where $CP_\ell(I, A)((f, \alpha), (g, \beta))$ is the equalizer of the pair of arrows in the top row and $Z_\ell(\mathbb{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta))$ is the equalizer of the other diagonal pair of arrows. Moreover, the diagram serially commutes. The vertical arrow is induced by the effect of the pseudofunctor $\mathbb{M}(I, -) : \mathbb{M} \to \mathscr{V}$-**Cat** on $\mathscr{V}$-enriched homs, and hence monic by Condition 1. It follows that there exists an isomorphism $CP_\ell(I, A)((f, \alpha), (g, \beta)) \to Z_\ell(\mathbb{M}(I, A))(\Phi(f, \alpha), \Phi(g, \beta))$. One can check that these isomorphisms are part of a $\mathscr{V}$-functor Φ, which, obviously, is fully faithful.

It only rests to prove that Φ is essentially surjective on objects. An object (f, γ) of $Z_\ell(\mathbb{M}(I, A))$ gives rise to a $\mathscr{V}$-natural transformation

$$\gamma'_h : p(A \otimes f)h \cong h * f \xrightarrow{\gamma_h} f * h \cong p(f \otimes A)h.$$

By Conditions 1 and 2, γ' is induced by a unique $\alpha : p(A \otimes f) \Rightarrow p(f \otimes A)$. The equalities (2) and (3) for the 2-cell α follow from the fact that (f, γ) is an object in the lax centre of $\mathbb{M}(I, A)$ and the fact that $\mathbb{M}(A^2, A) \to [\mathbb{M}(I, A^2), \mathbb{M}(I, A)]$ is fully faithful. Now observe that $\Phi(f, \alpha) = (f, \gamma)$. This shows that Φ is essentially surjective on objects. Finally, α is invertible if and only if γ is invertible, so that proof also applies to centres. $\square$

Recall from [4] that if A is a right autonomous pseudomonoid, with right dualization $\bar{d} : A^\vee \to A$, every map $f : I \to A$ has a right dual in the monoidal $\mathscr{V}$-category $\mathscr{M}(I, A)$. A right dual of f is given by $\bar{d}(f^*)^\vee$, where f^* is a right adjoint to f. Then the full subcategory $\mathrm{Map}\mathscr{M}(I, A)$ of $\mathscr{M}(I, A)$ is right autonomous (in the classical sense that it has right duals).

The following theorem applies to the case of promonoidal enriched categories. See Section 7.

Theorem 4.4. *In addition to the hypothesis above, assume the following: $\mathscr{V}$ is a complete and cocomplete monoidal closed category, $\mathscr{M}$ has all right liftings, $\mathscr{M}(I, A)$ has a dense sub-$\mathscr{V}$-category included in* $\mathrm{Map}\mathscr{M}(I, A)$ *and $\mathscr{M}(I, -) : \mathscr{M} \to$* **Cat** *reflects equivalences. If A is left autonomous, then the centre of A coincides with the lax centre whenever both exist.*

Proof. By Theorem 4.3, there exists an isomorphism as depicted below.

$$\begin{tikzcd}
\mathscr{M}(I, ZA) \arrow[r, "\simeq"] \arrow[d, "\mathscr{M}(I{,}z_c)"'] \arrow[dr, phantom, "\cong"] & Z(\mathscr{M}(I, A)) \arrow[d, hook] \\
\mathscr{M}(I, Z_\ell A) \arrow[r, "\simeq"] & Z_\ell(\mathscr{M}(I, A))
\end{tikzcd}$$

A straightforward modification of [6, Prop. 6] (using the property of the right liftings with respect to composition, dual to [24, Prop. 1]) shows that the monoidal $\mathscr{V}$-category $\mathscr{M}(I, A)$ is closed as a $\mathscr{V}$-category. It follows that the $\mathscr{V}$-functors $(f * -) = p(f \otimes A)- : \mathscr{M}(I, A) \to \mathscr{M}(I, A)$ given by tensoring with an object f are cocontinuous. As $\mathscr{M}(I, A)$ has a dense sub-monoidal $\mathscr{V}$-category with right duals, the hypotheses of [5, Theorem 3.4] are satisfied, and we deduce that the inclusion $Z(\mathscr{M}(I, A)) \hookrightarrow Z_\ell(\mathscr{M}(I, A))$ is the identity. It follows that $\mathscr{M}(I, z_c)$ is an equivalence, and hence z_c is an equivalence. $\square$

5. Hopf modules

In this section we recall the main results of [13] that will be used in next section study of lax centres of autonomous pseudomonoids.

We call 1-cells with right adjoint *maps*. Let A be a map pseudomonoid, *i.e.*, a pseudomonoid whose multiplication and unit are maps. There is a monad θ on $\mathscr{M}(A \otimes -, A)$ in the 2-category $[\mathscr{M}^{\mathrm{op}}, \mathbf{Cat}]$ given by $\theta_X(f) = p(A \otimes f)(p^* \otimes X)$. The unit and counit of the monad are induced by the counits of the adjunctions $j \dashv j^*$ and $p \dashv p^*$ respectively. An Eilenberg–Moore construction $\varphi \dashv \upsilon$ for θ is the analogue of the category of Hopf modules for a (coquasi)bialgebra. When the pseudonatural transformation $\lambda = \varphi\mathscr{M}(j^* \otimes -, A) : \mathscr{M}(-, A) \to \mathscr{M}(A \otimes -, A)^\theta$ is an equivalence we say that the *theorem of Hopf modules holds* for A. This is because when $\mathscr{M}$ is the bicategory of comodules and A is a coquasibialgebra we obtain the classical fundamental theorem of Hopf modules of Hopf algebra theory. See [13, Example 3.1].

The basic result of [13] is Theorem 6.2. Part of it states that a map pseudomonoid is left autonomous if and only if the theorem of Hopf modules holds. Other equivalent conditions are also given.

When the Gray monoid $\mathscr{M}$ is right closed, θ is birepresentable by a monad $t : [A, A] \to [A, A]$ in $\mathscr{M}$. As a 1-cell, t is the composition of the 1-cell $[A, A] \to [A^2, A^2]$ corresponding to $A \otimes \mathrm{ev}$ and $[p^*, p] : [A^2, A^2] \to [A, A]$. A *Hopf module construction* for A is defined as an Eilenberg–Moore construction for t. Hopf module constructions need not to exist, but A is left autonomous if and only if $(A^\circ \otimes p)(\mathrm{n} \otimes A) : A \to A^\circ \otimes A$ is a Hopf module construction for A (See [13, Theorem 6.5]).

6. Lax centres of autonomous pseudomonoids

In this section we exhibit the lax centre of a left autonomous map pseudomonoid as an Eilenberg–Moore construction for a certain monad. At the end of the section we compare the lax centre and the centre, a question also considered in [14].

The lax centre of a pseudomonoid was defined as a birepresentation of the pseudofunctor $CP_\ell(-, A)$. An object of the category $CP_\ell(X, A)$, *i.e.*, a lax centre piece, is a 2-cell $p(f \otimes A) \Rightarrow p(A \otimes f)c_{X,A}$ satisfying axioms. We observe that the same notion of lax centre can be defined by using c^* instead of c. In an entirely analogous way to Definition 3.1, one defines a category $CP^*_\ell(X, A)$ as follows. It has objects (f, γ) where $f : X \to A$ and $\gamma : p(f \otimes A)c^*_{X,A} \Rightarrow p(A \otimes f)$, and arrows $(f, \gamma) \to (g, \delta)$ those 2-cells $f \Rightarrow g$ which are compatible with γ and δ. Pasting with the canonical isomorphism $c_{X,A}c^*_{X,A} \cong 1_{X \otimes A}$ induces pseudonatural equivalences $CP_\ell(X, A) \to CP^*_\ell(X, A)$. This is the reason why the c^* appears in the following definition.

Definition 6.1. Given a map pseudomonoid A in a braided Gray monoid $\mathscr{M}$ define a pseudonatural transformation $\sigma : \mathscr{M}(A \otimes -, A) \Rightarrow \mathscr{M}(A \otimes -, A)$ with components

$$\sigma_X(g) = \left(A \otimes X \xrightarrow{p^* \otimes 1} A^2 \otimes X \xrightarrow{1 \otimes c^*_{X,A}} A \otimes X \otimes A \xrightarrow{g \otimes 1} A^2 \xrightarrow{p} A \right).$$

Lemma 6.2. *The pseudonatural transformation σ has a canonical structure of a monad.*

Proof. Just note that σ is isomorphic to the monad θ of Section 5 (see [13, Definition 3.1]) for the map pseudomonoid $(A, j, pc^*_{A,A})$. $\square$

Explicitly, the multiplication of σ is given by components

$$\begin{tikzcd}
A \otimes X \otimes A \arrow[r, "p^* \otimes 1 \otimes 1"] & A^2 \otimes X \otimes A \arrow[dr, bend left, "1 \otimes c^*_{X,A} \otimes 1"] \arrow[d, phantom, "\cong"] & & & & \\
A^2 \otimes X \arrow[u, "1 \otimes c^*_{X,A}"] \arrow[r, "p^* \otimes 1 \otimes 1"] \arrow[ur, phantom, "\cong"] & A^3 \otimes X \arrow[u, "A^2 \otimes c^*_{X,A}"'] \arrow[r, "1 \otimes c^*_{X,A^2}"] \arrow[dr, "1 \otimes p \otimes 1"] & A \otimes X \otimes A^2 \arrow[r, "g \otimes A^2"] \arrow[dr, "1 \otimes 1 \otimes p"] \arrow[d, phantom, "\cong"] & A^3 \arrow[r, "p \otimes 1"] \arrow[dr, "1 \otimes p"] \arrow[d, phantom, "\cong"] & A^2 \arrow[dr, "p"] \arrow[d, phantom, "\cong"] & \\
A \otimes X \arrow[u, "p^* \otimes 1"] \arrow[r, "p^* \otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \otimes X \arrow[u, "1 \otimes p^* \otimes 1"'] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow"] & A^2 \otimes X \arrow[r, "1 \otimes c^*_{X,A}"'] & A \otimes X \otimes A \arrow[r, "g \otimes 1"'] & A^2 \arrow[r, "p"'] & A
\end{tikzcd} \tag{7}$$

and the unit by

$$\begin{tikzcd}
& A \otimes X \arrow[r, equal] & A \otimes X \arrow[r, "g"] & A \arrow[dr, "1 \otimes j"] \arrow[drr, bend left, "1"] & & \\
A \otimes X \arrow[ur] \arrow[r, "p^* \otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \otimes X \arrow[u, "1 \otimes j^* \otimes 1"] \arrow[r, "1 \otimes c^*_{X,A}"'] \arrow[ur, phantom, "\cong"] & A \otimes X \otimes A \arrow[u, "1 \otimes 1 \otimes j^*"] \arrow[r, "g \otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \arrow[u, "1 \otimes j^*"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow"] & A^2 \arrow[r, "p"'] \arrow[u, phantom, "\cong"] & A
\end{tikzcd} \tag{8}$$

Now we assume that the braided Gray monoid $\mathscr{M}$ is also closed. In this situation the monads θ and σ are represented by monads t and $s : [A, A] \to [A, A]$. The monad s is

$$[A, A] \xrightarrow{i^A_A} [A \otimes A, A \otimes A] \xrightarrow{[c_{A,A}, c^*_{A,A}]} [A \otimes A, A \otimes A] \xrightarrow{[p^*, p]} [A, A], \tag{9}$$

which is the monad t for $(A, j, pc^*_{A,A})$. Alternatively, t and s can be taken respectively as

$$[A, A] \xrightarrow{\mathrm{id} \otimes 1} [A, A] \otimes [A, A] \longrightarrow [A \otimes A, A \otimes A] \xrightarrow{[p^*, p]} [A, A] \tag{10}$$

$$[A, A] \xrightarrow{1 \otimes \mathrm{id}} [A, A] \otimes [A, A] \longrightarrow [A \otimes A, A \otimes A] \xrightarrow{[p^*, p]} [A, A] \tag{11}$$

where $\mathrm{id} : I \to [A, A]$ is the 1-cell corresponding to 1_A under the equivalence $\mathscr{M}(A, A) \simeq \mathscr{M}(I, [A, A])$.

Observation 6.3. At this point we should remark that, for a map pseudomonoid A, $[A, A]$ has two pseudomonoid structures. The one we have considered so far is the *composition* pseudomonoid structure, but we also have the *convolution* pseudomonoid structure.

If (C, e, b) is a pseudocomonoid in the closed braided Gray monoid $\mathscr{M}$, $[C, -]$ is lax monoidal in the standard way. The unit constraint $I \to [C, I]$ corresponds under the closedness equivalence to the counit $e : C \to I$ and the 1-cells

$[C, X] \otimes [C, Y] \to [C, X \otimes Y]$ correspond to the composite

$$C \otimes [C, X] \otimes [C, Y] \xrightarrow{b\otimes 1\otimes 1} C^2 \otimes [C, X] \otimes [C, Y] \xrightarrow{1\otimes c\otimes 1} (C \otimes [C, X])^2 \xrightarrow{(\mathrm{ev}\otimes 1)(1\otimes 1\otimes \mathrm{ev})} X \otimes Y.$$

In particular, for a pseudomonoid A, $[C, A]$ has a canonical *convolution* pseudomonoid structure. This structure corresponds to the usual convolution tensor product in $\mathscr{M}(C, A)$ given by $f * g = p(A \otimes g)(f \otimes C)b$ with unit je. As remarked in [13, Observation 3.3], for a map pseudomonoid A the identity 1_A has a canonical structure of a monoid in the convolution monoidal category $\mathscr{M}(A, A)$. It follows that the corresponding 1-cell $\mathrm{id} : I \to [A, A]$ is a monoid in $\mathscr{M}(I, [A, A])$.

Observation 6.4. Let B be a pseudomonoid in $\mathscr{M}$ and consider $\mathscr{M}(I, B)$ and $\mathscr{M}(B, B)$ as monoidal categories with the convolution and the composition tensor product respectively. We have monoidal functors $L, R : \mathscr{M}(I, B) \to \mathscr{M}(B, B)$ given by $L(f) = p(f \otimes B)$ and $R(f) = p(B \otimes f)$. The associativity constraint of B induces isomorphisms $L(f)R(g) \cong R(g)L(f)$, natural in f and g. If m and n are monoids in $\mathscr{M}(I, B)$, then these isomorphisms form an invertible distributive law between the monads $L(m)$ and $R(n)$ on B.

The monoidal functors L, R are compatible with weak monoidal pseudofunctors (in the sense of [6]): if $F : \mathscr{M} \to \mathscr{N}$ is a weak monoidal pseudofunctor, then there are monoidal isomorphisms

$$\begin{tikzcd}
\mathscr{M}(I,B) \arrow[r, "{L,R}"] \arrow[d, "F_{I,B}"'] & \mathscr{M}(B,B) \arrow[dd, "F_{B,B}"] \\
\mathscr{N}(FI,FB) \arrow[d, "\simeq"'] \arrow[r, phantom, "\cong"] & \\
\mathscr{N}(I,FB) \arrow[r, "{L,R}"] & \mathscr{N}(FB,FB)
\end{tikzcd}$$

In particular, if m is a monoid in $M(I, B)$, we have isomorphisms $F(L(m)) \cong L(Fm)$ and $F(R(m)) \cong R(Fm)$ of monoids in $\mathscr{M}(B, B)$.

Proposition 6.5. *There exists an invertible distributive law between the monads t and s, and hence between the monads θ and σ.*

Proof. Apply the above Observation 6.4 to the *convolution* pseudomonoid $B = [A, A]$ and the monoid $m = n = \mathrm{id} : I \to [A, A]$, noting that $t = L(\mathrm{id})$ and $s = R(\mathrm{id})$. The 1-cell id is a monoid with the structure given by Observation 6.3. □

Denote by $\hat{\sigma}$ the lifting of σ to the Eilenberg–Moore object of θ. If t has an Eilenberg–Moore construction $u : [A, A]^t \to [A, A]$ the monad $\hat{\sigma}$ is represented by some $\hat{s} : [A, A]^t \to [A, A]^t$.

Proposition 6.6. *The monads s and $\hat{s}$ are opmonoidal monads.*

Proof. As we noted above, s is the monad t corresponding to the pseudomonoid $(A, j, pc^*_{A,A})$. By [13, Proposition 5.1], t is an opmonoidal monad, with respect to the composition monoidal structure of $A^\circ \otimes A$. It follows that s is opmonoidal too. The monad $\hat{s}$ is opmonoidal since $[A, A]^t$ is an Eilenberg–Moore construction in **Opmon**$(\mathscr{M})$. □

Sometimes the pseudonatural transformation $\hat{\sigma} : \mathscr{M}(A\otimes -, A)^\theta \to \mathscr{M}(A\otimes -, A)^\theta$ defined above can be restricted along the pseudonatural transformation λ of Section 5. Suppose that there exists a pseudonatural transformation $\tilde{\sigma} : \mathscr{M}(-, A) \to \mathscr{M}(-, A)$ such that $\lambda\tilde{\sigma} \cong \hat{\sigma}\lambda$; since λ is fully faithful (see [13, Proposition 3.6]), this is equivalent to saying that for each X the monad $\hat{\sigma}_X$ restricts to a monad on the replete image of λ_X in $\mathscr{M}(A\otimes X, A)^{\theta_X}$, and in this case $\tilde{\sigma} = \lambda^*\hat{\sigma}\lambda$. Moreover, $\tilde{\sigma}$ carries the structure of a monad induced by that of $\hat{\sigma}$, making λ together with the isomorphism $\lambda\tilde{\sigma} \cong \hat{\sigma}\lambda$ a monad morphism. Such a monad $\tilde{\sigma}$ clearly exists if the theorem of Hopf modules holds for A, *i.e.*, if λ is an equivalence.

Theorem 6.7. *There exists an equivalence in the 2-category* **Hom**$(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})$ *between $\mathscr{M}(-, A)^{\tilde{\sigma}}$ and $CP_\ell(-, A)$ whenever the monad $\tilde{\sigma}$ exists. Moreover, this equivalence commutes with the corresponding forgetful pseudonatural transformations into $\mathscr{M}(-, A)$.*

Proof. Instead of $\tilde{\sigma}_X$, we shall consider the restriction of $\hat{\sigma}_X$ to the replete image of λ_X. Take $f : X \to A$ and assume that $\lambda_X(f : X \to A)$ has a structure ν of $\hat{\sigma}$-algebra. This means that the action ν is a 2-cell

$$\begin{tikzcd}
A\otimes A\otimes X \arrow[rr, "1\otimes c^*_{X,A}"] & & A\otimes X\otimes A \arrow[d, "1\otimes f\otimes 1"] \\
A\otimes X \arrow[u, "p^*\otimes 1"] \arrow[d, "1\otimes f"'] \arrow[rr, phantom, "\overset{\Leftarrow}{\nu}"] & & A\otimes A\otimes A \arrow[d, "p\otimes 1"] \\
A\otimes A \arrow[dr, "p"'] & & A\otimes A \arrow[dl, "p"] \\
& A &
\end{tikzcd} \tag{12}$$

which is a morphism of θ_X-algebras from $\hat{\sigma}_X\lambda_X(f)$ to $\lambda_X(f)$. Furthermore, the pasting

$$\begin{tikzcd}
 & & A^2\otimes X\otimes A \arrow[rr, "1\otimes c^*\otimes 1"] & & A\otimes X\otimes A^2 \arrow[d, "1\otimes f\otimes A^2"] \\
A^2\otimes X \arrow[rr, "1\otimes c^*"] & & A\otimes X\otimes A \arrow[u, "p^*\otimes 1\otimes 1"] \arrow[d, "1\otimes f\otimes 1"] \arrow[rr, phantom, "\Leftarrow \nu\otimes 1"] & & A^4 \arrow[d, "p\otimes A^2"] \\
A\otimes X \arrow[u, "p^*\otimes 1"] \arrow[d, "1\otimes f"] & & A^3 \arrow[dr, "p\otimes 1"'] & & A^3 \arrow[dl, "p\otimes 1"] \\
A^2 \arrow[drr, "p"'] \arrow[rrr, phantom, "\Leftarrow \nu"] & & & A^2 \arrow[dl, "p"] & \\
 & & A & &
\end{tikzcd}$$

is equal to the composite $\sigma_X\sigma_X\lambda_X(f) \to \sigma_X\lambda_X(f) \xrightarrow{\nu} \lambda_X(f)$ of the multiplication of σ_X (7) and ν, and the composition $\lambda_X(f) \to \sigma_X\lambda_X(f) \xrightarrow{\nu} \lambda_X(f)$ of the unit of σ (8) and ν is the identity. The 2-cells (12) correspond, under pasting with $\phi^{-1} : p(A\otimes p) \cong p(p\otimes A)$, to 2-cells $p(A\otimes(p(f\otimes A)c^*_{X,A}))(p^*\otimes X) \Rightarrow p(A\otimes f)$, and then to 2-cells $p(A\otimes(p(f\otimes A)c^*_{X,A})) \Rightarrow p(A\otimes f)(p\otimes A) \cong p(A\otimes p)(A\otimes A\otimes f)$. Since λ_X is fully faithful, and $\hat{\sigma}$ restricts to the replete image of λ_X, it follows that the 2-cells ν correspond to the 2-cells γ as in (1). The axiom of associativity for the action ν translates into the axiom (2) for γ and the axiom of unit for ν into the axiom (3) for γ. This shows that the composition of the forgetful functor $V_X : CP_\ell(X,A) \to \mathscr{M}(X,A)$ with λ_X factors as a pseudonatural transformation G followed by $\hat{U}$, as depicted below.

$$\begin{tikzcd}
CP_\ell(X,A) \arrow[rr, dashed, "H_X"] \arrow[d, "V_X"'] \arrow[drr, "G_X"' near start] & & \mathscr{M}(X,A)^{\tilde{\sigma}_X} \arrow[d, "\tilde{\lambda}_X"] \arrow[dll, "\tilde{U}_X" near start] \\
\mathscr{M}(X,A) \arrow[dr, "\lambda_X"'] & & (\mathscr{M}(A\otimes X,A)^{\theta_X})^{\hat{\sigma}_X} \arrow[dl, "\hat{U}_X"] \\
 & \mathscr{M}(A\otimes X,A)^{\theta_X} &
\end{tikzcd}$$

Moreover, G_X factors through the image of $\tilde{\lambda}_X$, since $\hat{U}_XG_X$ factors through λ_X, and in fact G_X is an equivalence into the image of $\tilde{\lambda}_X$. Here $\tilde{\lambda}_X$ is the functor induced on Eilenberg–Moore constructions by λ_X; in particular, $\tilde{\lambda}_X$ is fully faithful since λ_X is fully faithful. Therefore we have an equivalence H_X as in the diagram, such that $\tilde{\lambda}_XH_X = G_X$. Hence, $\lambda_X\tilde{U}_XH_X = \hat{U}_X\tilde{\lambda}_XH_X = \hat{U}_XG_X = \lambda_XV_X$, and $\tilde{U}_XH_X = V_X$. The equivalences H_X are clearly pseudonatural in X. $\square$

Corollary 6.8. *If the theorem of Hopf modules holds for a map pseudomonoid A then there exists an equivalence $CP_\ell(-,A) \simeq \mathscr{M}(A\otimes -,A)^{\sigma\theta}$.*

Proof. λ_X is an equivalence and then the monad $\tilde{\sigma}$ exists and

$$\mathscr{M}(-,A)^{\tilde{\sigma}} \simeq (\mathscr{M}(A\otimes -,A)^{\theta})^{\hat{\sigma}} \simeq \mathscr{M}(A\otimes -,A)^{\sigma\theta}. \quad \square$$

Following [13], we call $\ell : A \to [A,A]^t$ the 1-cell that birepresents λ.

Theorem 6.9. *Suppose that the theorem of Hopf modules holds for the map pseudomonoid A and that it has a Hopf module construction. Then a lax centre of A is an Eilenberg–Moore construction for the opmonoidal monad*

$$\tilde{s} := \ell^*\hat{s}\ell = A \to A$$

one of them existing if the other does. Moreover,

$$\tilde{s} \cong \left(A \xrightarrow{j\otimes 1} A\otimes A \xrightarrow{p^*\otimes 1} A\otimes A\otimes A \xrightarrow{1\otimes c^*_{A,A}} A\otimes A\otimes A \xrightarrow{p\otimes A} A\otimes A \xrightarrow{p} A \right).$$

Proof. The monad $\hat{s}$ exists and is opmonoidal since $t : [A,A] \to [A,A]$ has an Eilenberg–Moore construction in **Opmon**($\mathscr{M}$). Hence, $\tilde{s}$ has a canonical opmonoidal monad structure induced by the one of $\hat{s}$. Theorem 6.7 implies that the lax centre of A exists, that is, $CP(-,A)$ is birepresentable, if and only if the monad $\tilde{s}$ has an Eilenberg–Moore construction.

To obtain an expression for the 1-cell $\tilde{s}$ recall that, by definition, $\mathscr{M}(-, \tilde{s})$ is isomorphic to $\lambda^* \hat{\sigma} \lambda$. It is easy to show that

$$\begin{aligned} \lambda_X^* \hat{\sigma}_X \lambda_X (f : X \to A) &= p(p \otimes A)(A \otimes f \otimes A)(A \otimes c^*_{X,A})(p^* \otimes X)(j \otimes X) \\ &\cong p(p \otimes A)(A \otimes c^*_{A,A})(p^* \otimes A)(j \otimes A) f; \end{aligned}$$

see the definition of λ in Section 5 and Definition 6.1. It follows that the expression for $\tilde{s}$ of the statement holds. □

Observation 6.10. The thesis of Theorem 6.9 above holds under the sole hypothesis of A being left autonomous. This is so because every left autonomous map pseudomonoid has a Hopf module construction. See Section 5.

Now we concentrate in the case of autonomous map pseudomonoids. Let A be such a pseudomonoid. The internal hom $[A, A]$ is given by $A^\circ \otimes A$, where A° is a right bidual of A.

Corollary 6.11. *Suppose $F : \mathscr{M} \to \mathscr{N}$ is a pseudofunctor between Gray monoids with the following properties: F preserves Eilenberg–Moore objects, is braided and strong monoidal. Then, F preserves lax centres of left autonomous map pseudomonoids.*

Proof. Let A be a left autonomous map pseudomonoid in $\mathscr{M}$. By Observation 6.10, the lax centre of A is the Eilenberg–Moore construction for the opmonoidal monad $\tilde{s} : A \to A$, one existing if the other does. On the other hand, FA is also a left autonomous map pseudomonoid by [13, Proposition 6.10]. Therefore, it is enough to show that F preserves the monad $\tilde{s}$, in the sense that $F\tilde{s}$ is isomorphic to the corresponding monad $\tilde{s}$ for FA.

Since $\tilde{s}$ is the lifting of the monad s on $A^\circ \otimes A$ to the Eilenberg–Moore construction $(A^\circ \otimes p)(\mathrm{n} \otimes A) : A \to A^\circ \otimes A$ of the monad t (see Section 5) it suffices to prove that F preserves the monads t and s. We only work with t, the proof for the monad s being completely analogous. We know from the proof of Proposition 6.5 that $t = L(\mathrm{n})$ and $s = R(\mathrm{n})$, where $L, R : \mathscr{M}(I, A^\circ \otimes A) \to \mathscr{M}(A^\circ \otimes A, A^\circ \otimes A)$ are the functors defined in Observation 6.4. Therefore, $Ft = F(L(\mathrm{n})) \cong L(I \xrightarrow{\cong} FI \xrightarrow{F\mathrm{n}_A} F(A^\circ \otimes A)) \cong L(\mathrm{n}_{FA})$, which is the monad t corresponding to the pseudomonoid FA. □

Theorem 6.12. *For a (left and right) autonomous map pseudomonoid the centre equals the lax centre, either existing if the other does.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
(\mathscr{M}(A\otimes X, A)^{\theta_X})^{\hat{\sigma}_X} \arrow[r] \arrow[d] & \mathscr{M}(A\otimes X, A)^{\theta_X} \arrow[r, "\hat{\sigma}"] \arrow[d] & \mathscr{M}(A\otimes X, A)^{\theta_X} \arrow[d, "\upsilon_X"] \\
\mathscr{M}(A\otimes X, A)^{\sigma_X} \arrow[r] & \mathscr{M}(A\otimes X, A) \arrow[r, "\sigma"'] & \mathscr{M}(A\otimes X, A)
\end{tikzcd}$$

In Theorem 6.7 we proved that any lax centre piece arises as

$$\begin{tikzcd}
A\otimes A\otimes X \arrow[r, equal] \arrow[dr, "p\otimes 1"'] & A\otimes A\otimes X \arrow[rr, "1\otimes c_{A,X}"] \arrow[d, phantom, "\Downarrow", near start] & & A\otimes X\otimes A \arrow[d, "h\otimes 1"] \\
& A\otimes X \arrow[u, "p^*\otimes 1"'] \arrow[dr, "h"'] & \arrow[d, phantom, "\Leftarrow \nu"] & A\otimes A \arrow[dl, "p"] \\
& & A &
\end{tikzcd} \tag{13}$$

for some $\hat{\sigma}_X$-algebra $\nu : \hat{\sigma}_X(h) \to h$, so we have to prove that (13) is invertible. Consider the canonical split coequalizer $\hat{\sigma}_X^2(h) \rightrightarrows \hat{\sigma}_X(h) \twoheadrightarrow h$ in $\mathscr{M}(A \otimes X, A)^{\theta_X}$, and its image $\nu : \sigma_X(h) \to h$ in $\mathscr{M}(A \otimes X, A)$. The arrow ν is a morphism of σ_X-algebras. This implies that the lower rectangle in the diagram below commutes.

$$\begin{tikzcd}
p(\sigma_X(h)\otimes A)(A\otimes c_{A,X}) \arrow[r] \arrow[d, "p(\sigma_X(h)\otimes A)(A\otimes c_{A,X})(\eta\otimes X)"] & p(h\otimes A)(A\otimes c_{A,X}) \arrow[d, "p(h\otimes A)(A\otimes c_{A,X})(\eta\otimes X)"] \\
p(\sigma_X(h)\otimes A)(A\otimes c_{A,X})(p^* p\otimes X) \arrow[d, equal] & p(h\otimes A)(A\otimes c_{A,X})(p^* p\otimes X) \arrow[d, equal] \\
\sigma_X^2(h)(p\otimes X) \arrow[r] \arrow[d, "(\mu_X)_h(p\otimes X)"] & \sigma_X(h)(p\otimes X) \arrow[d, "\nu(p\otimes X)"] \\
\sigma_X(h)(p\otimes X) \arrow[r] & h(p\otimes X)
\end{tikzcd}$$

The upper rectangle commutes by naturality of composition. Here η denotes the unit of the adjunction $p \dashv p^*$ and μ the multiplication of the monad σ. Observe that the rows are coequalizers and the right-hand column is just (13). Then, to show that this last arrow is invertible it suffices to show that the left-hand side column, which is the pasting of η with the multiplication of σ (7), is so. But this 2-cell is invertible because A is right autonomous and hence by the dual of [13, Theorem 6.4 (ii)] 2-cell below is invertible. This completes the proof.

$$\begin{tikzcd}
A^2 \arrow[rr, equal] \arrow[dr, "p"'] & & A^2 \arrow[r, "p^*\otimes 1"] & A^3 \arrow[dr, "1\otimes p"] \arrow[d, phantom, "\Downarrow"] & \\
& A \arrow[ur, "p^*"'] \arrow[r, "p^*"'] \arrow[u, phantom, "\Downarrow"] & A^2 \arrow[ur, "1\otimes p^*"'] \arrow[rr, equal] \arrow[u, phantom, "\cong"] & & A^2
\end{tikzcd}
=
\begin{tikzcd}
& A^3 \arrow[r, "1\otimes p"] \arrow[dr, "p\otimes 1"] \arrow[d, phantom, "\Downarrow"] & A^2 \arrow[rr, equal] \arrow[dr, "p"'] \arrow[d, phantom, "\cong"] & & A^2 \\
A^2 \arrow[ur, "p^*\otimes 1"] \arrow[rr, equal] & & A^2 \arrow[r, "p"'] & A \arrow[ur, "p^*"'] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd}
\qquad \square$$

Finally, putting together the results above we obtain the following corollary. Compare with [14].

Corollary 6.13. *Any autonomous map pseudomonoid in a braided monoidal bicategory with Eilenberg–Moore objects has both a centre and a lax centre, and the two coincide.*

7. Enriched (pro)monoidal categories

In this section we interpret the results of the previous section in the particular context of the bicategory of $\mathscr{V}$-modules.

7.1. Review of the bicategory of $\mathscr{V}$-modules

In this section we give the barest recount of the bicategory of $\mathscr{V}$-modules. We use the same conventions as in [13], where the reader can find more details.

Let $\mathscr{V}$ be a complete and cocomplete closed symmetric monoidal category. The bicategory $\mathscr{V}$-**Mod** has small $\mathscr{V}$-categories as objects and the $\mathscr{V}$-functor categories $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B}) = [\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B},\mathscr{V}]_0$ has hom-categories. 1-cells in $\mathscr{V}$-**Mod** are called $\mathscr{V}$-modules. The composite of two $\mathscr{V}$-modules $M : \mathscr{A} \to \mathscr{B}$ and $N : \mathscr{B} \to \mathscr{C}$ is given by $(NM)(a, c) = \int^x N(x, c) \otimes M(a, x)$. The identity module $1_{\mathscr{A}}$ is given by $1_{\mathscr{A}}(a, a') = \mathscr{A}(a, a')$.

The tensor product of $\mathscr{V}$-categories induces a structure of a monoidal bicategory on $\mathscr{V}$-**Mod**. Moreover, the usual symmetry of $\mathscr{V}$-**Cat** together with the symmetry of $\mathscr{V}$ induce a structure of symmetric monoidal bicategory on $\mathscr{V}$-**Mod**; more precisely, there is a symmetric Gray monoid (described in [6, pp. 138–139]) canonically monoidally biequivalent to $\mathscr{V}$-**Mod**, and such that its symmetry corresponds to the "symmetry" in $\mathscr{V}$-**Mod** described above. There is a pseudofunctor $(-)_* : \mathscr{V}\text{-}\mathbf{Cat}^{\mathrm{co}} \to \mathscr{V}\text{-}\mathbf{Mod}$ which is the identity on objects and on hom-categories $[\mathscr{A},\mathscr{B}]_0^{\mathrm{op}} \to [\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B},\mathscr{V}]_0$ sends a $\mathscr{V}$-functor F to the $\mathscr{V}$-functor $F_*(a, b) = \mathscr{B}(F(a), b)$. Moreover, the $\mathscr{V}$-module F_* has right adjoint F^* given by $F^*(b, a) = \mathscr{B}(b, F(a))$. The pseudofunctor $(-)_*$ is easily shown to be strong monoidal and symmetry-preserving.

Each object $\mathscr{A}$ has a left and right bidual provided by the opposite $\mathscr{V}$-category $\mathscr{A}^{\mathrm{op}}$.

One of the many pleasant properties of $\mathscr{V}$-**Mod** is that it has right liftings. If $M : \mathscr{B} \to \mathscr{C}$ and $N : \mathscr{A} \to \mathscr{C}$ are $\mathscr{V}$-modules, a right lifting of N through M is given by the formula ${}^{M}N(a, b) = \int_{c\in\mathscr{C}}[M(b, c), N(a, c)]$. As explained in Section 4, the existence of right liftings endows each hom-category $\mathscr{V}\text{-}\mathbf{Mod}(I,\mathscr{A})$ with a canonical structure of a $\mathscr{V}\text{-}\mathbf{Mod}(I, I)$-category, where I is the trivial $\mathscr{V}$-category. Therefore, each $\mathscr{V}\text{-}\mathbf{Mod}(I,\mathscr{A})$ is canonically a $\mathscr{V}$-category via the monoidal isomorphism $\mathscr{V}\text{-}\mathbf{Mod}(I, I) \cong \mathscr{V}$. This is exactly the usual $\mathscr{V}$-category structure of $[\mathscr{A},\mathscr{V}]$. In fact, each hom-category $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B})$ is canonically a $\mathscr{V}$-category, in a way such that the equivalence $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B}) \simeq \mathscr{V}\text{-}\mathbf{Mod}(I,\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B})$ is a $\mathscr{V}$-functor.

Another feature of $\mathscr{V}$-**Mod** we will need is the existence of Kleisli and Eilenberg–Moore constructions for monads. The existence of the former was shown in [23]. Here we recall the explicit construction for later use. If (M, η, μ) is a monad in $\mathscr{V}$-**Mod** on $\mathscr{A}$, $\mathsf{Kl}(M)$ has the same objects as $\mathscr{A}$ and homs $\mathsf{Kl}(M)(a, b) = M(a, b)$. Composition is given by

$$M(b, c)\otimes M(a, b) \to \int^{b\in\mathscr{A}} M(b, c)\otimes M(a, b) \xrightarrow{\mu_{a,c}} M(a, c)$$

and the units by $I \xrightarrow{\mathrm{id}} \mathscr{A}(a, a) \xrightarrow{\eta_{a,a}} M(a, a)$. One can verify that the $\mathscr{V}$-module K_* induced by the $\mathscr{V}$-functor $K : \mathscr{A} \to \mathsf{Kl}(M)$ given by the identity on objects and by $\eta_{a,b} : \mathscr{A}(a, b) \to M(a, b)$ on homs has the universal property of the Kleisli construction. It is not hard to see that K^* is an Eilenberg–Moore construction for M.

7.2. Lax centres in $\mathscr{V}$-**Mod**

In this section we study the centre and lax centre of pseudomonoids in the monoidal bicategory of $\mathscr{V}$-modules by means of the theory developed in previous sections. Along the way, we compare our work with [5,7].

First we consider lax centres of arbitrary pseudomonoids. We shall show that the results in Section 4 apply to $\mathscr{V}$-**Mod**. To realise this aim, we have to verify all the hypotheses required in that section.

We already saw in Section 7.1 that right liftings exist. In order to show $\mathscr{V}$-**Mod** satisfies Conditions 1 and 2 in Section 4 it is enough to prove that the arrow (6) is an isomorphism for $\mathscr{M}$ the bicategory of $\mathscr{V}$-modules. In this case (6) becomes

$$[\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B},\mathscr{V}](M, N) \to [[\mathscr{A},\mathscr{V}],[\mathscr{B},\mathscr{V}]]((M\circ -), (N\circ -)), \tag{14}$$

where $(M \circ -)$ is the $\mathscr{V}$-functor given by composition with the $\mathscr{V}$-module M. To show that (14) is an isomorphism, recall that the $\mathscr{V}$-functor

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}] \cong [\mathscr{B}, [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]] \to \mathrm{Cocts}[[\mathscr{B}^{\mathrm{op}}, \mathscr{V}], [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]] \tag{15}$$

into the $\mathscr{V}$-category of cocontinuous $\mathscr{V}$-functors is an equivalence by [12, Theorem 4.51]. This $\mathscr{V}$-functor sends $R : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathscr{V}$ to the left extension of the corresponding $R' : \mathscr{B} \to [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ along the Yoneda embedding $\mathrm{y} : \mathscr{B} \to [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$, which is exactly $(R \circ -)$.

Theorem 4.3 gives:

Corollary 7.1. *Suppose the lax centre of the promonoidal $\mathscr{V}$-category $\mathscr{A}$ exists. Then there exits an equivalence of $\mathscr{V}$-categories $[Z_\ell\mathscr{A}, \mathscr{V}] \simeq Z_\ell[\mathscr{A}, \mathscr{V}]$, where on the left-hand side appears the lax centre in $\mathscr{V}$-**Mod** and on the right-hand side the lax centre in $\mathscr{V}$-**Cat**. The composition of this equivalence with the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{A}, \mathscr{V}] \to [\mathscr{A}, \mathscr{V}]$ is canonically isomorphic to the $\mathscr{V}$-functor given by composing with the universal $\mathscr{V}$-module $Z_\ell\mathscr{A} \to \mathscr{A}$. If the centre of $\mathscr{A}$, rather than the lax centre, exists, then the above holds substituting lax centres by centres throughout.*

Now we turn our attention to autonomous pseudomonoids. The existence of Eilenberg–Moore constructions in $\mathscr{V}$-**Mod** together with Theorems 6.9 and 6.12 imply:

Proposition 7.2. *Any left autonomous map pseudomonoid in $\mathscr{V}$-**Mod** has a lax centre. Moreover, if the pseudomonoid is also right autonomous then the lax centre is the centre.*

Proposition 7.3. *If a left autonomous pseudomonoid $\mathscr{A}$ in $\mathscr{V}$-**Mod** has a centre construction, then its lax centre and its centre coincide.*

Proof. We saw that the lax centre of a $\mathscr{A}$ exists. The result, then, follows from Theorem 4.4. The category $\mathscr{V}$-**Mod**$(I, \mathscr{A})$ has a dense small sub-$\mathscr{V}$-category, namely the one determined by the representable $\mathscr{V}$-functors; and representables are maps in the bicategory of $\mathscr{V}$-modules. The rest of the hypotheses on $\mathscr{M}$ required in Theorem 4.4 are easily verified. □

We shall describe the lax centre explicitly. In order to simplify the description, we will suppose $\mathscr{A}$ is a left autonomous monoidal $\mathscr{V}$-category, and not merely a promonoidal one. However, all the following description carries over to the case of map pseudomonoids.

By Theorem 6.9, the lax centre of $\mathscr{A}$ in $\mathscr{V}$-**Mod** is the Eilenberg–Moore construction for the monad $\tilde{S}$ given by

$$\mathscr{A} \xrightarrow{J\otimes 1} \mathscr{A} \otimes \mathscr{A} \xrightarrow{P^*\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1\otimes c_*} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{P\otimes 1} \mathscr{A} \otimes \mathscr{A} \xrightarrow{P} \mathscr{A} \tag{16}$$

where c denotes the usual symmetry in $\mathscr{V}$-**Cat**. Explicitly,

$$\tilde{S}(a; b) \cong \int^{x,y} \mathscr{A}(y \otimes (a \otimes x), b) \otimes \mathscr{A}(I, y \otimes x) \cong \int^{y} \mathscr{A}(y \otimes (a \otimes y^\vee), b),$$

where $y^\vee$ denotes the left dual of y in $\mathscr{A}$. The multiplication of this monad is given by

$$\begin{aligned} \tilde{S}^2(a; b) &\cong \int^{u,y,z} \mathscr{A}\left(y \otimes (u \otimes y^\vee), b\right) \otimes \mathscr{A}\left(z \otimes (a \otimes z^\vee), u\right) \\ &\cong \int^{y,z} \mathscr{A}\left(y \otimes (z \otimes (a \otimes z^\vee)) \otimes y^\vee, b\right) \cong \int^{y,z} \mathscr{A}\left((y \otimes z) \otimes (a \otimes (y \otimes z)^\vee), b\right) \to \\ &\longrightarrow \int^{x} \mathscr{A}\left(x \otimes (a \otimes x^\vee), b\right) \cong \tilde{S}(a; b) \end{aligned} \tag{17}$$

where the last arrow is induced by the components $\zeta^{a,b}_{y\otimes z} : \mathscr{A}\left((y \otimes z) \otimes (a \otimes (y \otimes z)^\vee), b\right) \to \int^x \mathscr{A}\left(x \otimes (a \otimes x^\vee), b\right)$ of the universal dinatural transformation defining the coend in the codomain of (17). The unit of $\tilde{S}$ is given by components $\zeta^{a,b}_I : \mathscr{A}(a, b) \to \int^x \mathscr{A}\left(x \otimes (a \otimes x^\vee), b\right)$ corresponding to $x = I$ of the same dinatural transformation. Now we have all the ingredients to describe the lax centre $Z_\ell(\mathscr{A})$, that is, a Kleisli construction for $\tilde{S}$. It has the same objects as $\mathscr{A}$, enriched homs $Z_\ell(\mathscr{A})(a, b) = \tilde{S}(a, b)$, composition given by the multiplication, and unit given by

$$I \to \mathscr{A}(a, a) \xrightarrow{\zeta^{a,a}_I} \tilde{S}(a, a),$$

where the first arrow is the identity of a in $\mathscr{A}$. The arrows $\zeta^{a,b}_I : \mathscr{A}(a, b) \to \tilde{S}(a, b)$ define a $\mathscr{V}$-functor, which we also call ζ, and the universal $Z_\ell(\mathscr{A}) \to \mathscr{A}$ is none other than ζ^*.

Observation 7.4. The monad $\tilde{S}$ is closely related to the monad $\check{M}$ in [7, Section 5]. There the authors show that for a general small promonoidal $\mathscr{V}$-category $\mathscr{C}$ there exists a monad $\check{M}$ on $\mathscr{C}$ in $\mathscr{V}$-**Mod** with the following property. Whenever $[\mathscr{C}, \mathscr{V}]$ has a small dense sub-$\mathscr{V}$-category of objects with left duals (it is *right-dual controlled*, in the terminology of [7]), the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{C}, \mathscr{V}] \to [\mathscr{C}, \mathscr{V}]$ is an Eilenberg–Moore construction for the monad M on $[\mathscr{C}, \mathscr{V}]$ in $\mathscr{V}$-**Cat** given by composition with $\check{M}$. The module $\check{M}$ is given by

$$\check{M}(a,b) = \int^{x,y} P(P \otimes \mathscr{C})(y,a,x,b) \otimes x^{\wedge}(y),$$

where $x^{\wedge}$ is the internal hom $[\![\mathscr{C}(x,-), J]\!] \in [\mathscr{C}, \mathscr{V}]$ (J is the unit of the promonoidal structure).

When $\mathscr{C}$ is equipped with a left dualization $D : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, each $\mathscr{V}$-module $I \to \mathscr{C}$ with right adjoint in $\mathscr{V}$-**Mod** has a left dual in the monoidal $\mathscr{V}$-category $\mathscr{V}$-**Mod**$(I, \mathscr{C}) = [\mathscr{C}, \mathscr{V}]$. This was first shown in [4]. In particular, $\mathscr{C}(x,-)$, which is the $\mathscr{V}$-module induced by the $\mathscr{V}$-functor $I \to \mathscr{C}$ constant on x, has left dual. It follows that $[\mathscr{C}, \mathscr{V}]$ has a small dense sub-$\mathscr{V}$-category with left duals, and the results of [7] mentioned above apply.

In this situation, if we assume J is a map, so that $\tilde{S}$ exists, we claim that the monads $\check{M}$ and $\tilde{S}$ are isomorphic, or more precisely, that they are isomorphic as monoids in the monoidal $\mathscr{V}$-category $\mathscr{V}$-**Mod**$(\mathscr{C}, \mathscr{C}) = [\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}, \mathscr{V}]$. To show this, it is enough to prove that the monads $(\check{M} \circ -)$ and $(\tilde{S} \circ -)$ on $\mathscr{V}$-**Mod**$(I, \mathscr{C}) = [\mathscr{C}, \mathscr{V}]$ given by composition with $\check{M}$ and $\tilde{S}$ respectively are isomorphic. Now, the monad $(\tilde{S} \circ -)$ is $\mathscr{V}$-**Mod**$(I, \tilde{S})$, and then it has the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{C}, \mathscr{V}] \to [\mathscr{C}, \mathscr{V}]$ as a (bicategorical) Eilenberg–Moore construction by Corollary 7.1 and Proposition 7.2. Then, $(\tilde{S} \circ -)$ and $M = (\check{M} \circ -)$ have the same Eilenberg–Moore construction in $\mathscr{V}$-**Cat** and it follows that both monads are isomorphic as required.

Example 7.5. Let $\mathscr{G}$ be a groupoid. Write $\Delta : \mathscr{G} \to \mathscr{G} \times \mathscr{G}$ for the diagonal functor and $E : \mathscr{G} \to 1$ the only possible functor. These give $\mathscr{G}$ a structure of comonoid in **Cat** and thus $P = \Delta^*$ and $J = E^*$ is a promonoidal structure on $\mathscr{G}$. Explicitly, $P(a,b;c) = \mathscr{G}(a,c) \times \mathscr{G}(b,c)$ and $J(a) = 1$; the monoidal structure induced in $[\mathscr{G}, \mathbf{Set}]$ is given by the point-wise cartesian product. Define a functor $D : \mathscr{G}^{\mathrm{op}} \to \mathscr{G}$ as the identity on objects and $D(f) = f^{-1}$ on arrows. In [6, Example 10] it was essentially shown that D is a left and right dualization for the map pseudomonoid $(\mathscr{G}, J, P)$ in **Set**-**Mod**$^{\mathrm{co}}$. Then, by Corollary 6.13, $\mathscr{G}$ has centre and lax centre in **Set**-**Mod**$^{\mathrm{co}}$ and both coincide. On the other hand, $[Z(\mathscr{G}), \mathbf{Set}] \simeq Z([\mathscr{G}, \mathbf{Set}])$ by Theorem 4.3, which together with [5, Theorem 4.5] shows that the centre of $\mathscr{G}$ in **Set**-**Mod** is equivalent to the category called (lax) centre of $\mathscr{G}$ in [5].

8. Comodules

This section deals with the case of the monoidal bicategories of comodules **Comod**$(\mathscr{V})$. In general, $\mathscr{V}$ will be a braided monoidal category with a certain completeness condition. However, when we consider the lax centre of pseudomonoids the braiding will be a symmetry. Our aim is to show how the general theory developed in previous sections specialises to some of the most basic results of the theory of Hopf algebras.

8.1. Background on the bicategory of comodules

If $\mathscr{V}$ is a monoidal category, the category of comonoids in $\mathscr{V}$, denoted by **Comon**$(\mathscr{V})$, has objects and 1-cells comonoids and comonoid morphisms in $\mathscr{V}$, respectively. 2-cells between 1-cells $C \to D$ are arrows $C \to I$ in $\mathscr{V}$ satisfying one axiom. See [13, Example 2.3] or [4].

When $\mathscr{V}$ has equalizers of reflexive pairs and these are preserved by each functor $(- \otimes X)$, $(X \otimes -)$, one can construct the bicategory of comodules in $\mathscr{V}$. This has been considered by several authors; see [4,8] for example. Sometimes, when $\mathscr{V}$ has further completeness properties, the bicategory of $\mathscr{V}$-comodules can be considered; see [6].

The bicategory of comodules in $\mathscr{V}$ has comonoids in $\mathscr{V}$ as objects, C–D-bicomodules as 1-cells $C \to D$ and bicomodule morphisms as 2-cells. Vertical composition is just composition of bicomodule morphisms. Horizontal composition is given by *cotensor product*; if $M : C \to D$ and $N : D \to E$ are bicomodules, its composition is given by the cotensor product $M \square_D E$, the equalizer of the obvious pair of arrows $M \otimes N \rightrightarrows M \otimes D \otimes N$ constructed with the coactions of M and N. See [13, Example 2.3] or [4] for details.

There is a pseudofunctor $(-)_* : \mathbf{Comon}(\mathscr{V}) \to \mathbf{Comod}(\mathscr{V})$ sending $f : C \to D$ to the bicomodule $f_* : C \to D$, which is C with the regular coaction corestricted by f on the right. Each comodule of the form f_* has a right adjoint.

The following well-known fact is useful.

Lemma 8.1. *Let C, D be two comonoids in $\mathscr{V}$, with counits ε_C, ε_D, respectively. A comodule $M : C \to D$ is isomorphic to f_* for some comonoid morphism $f : C \to D$ if and only if $(\varepsilon_D)_* M \cong (\varepsilon_C)_*$.*

Proof. We only give a sketch of a proof. Let $\alpha : (\varepsilon_D)_* M \to (\varepsilon_C)_*$ be an isomorphism of comodules. We define f as the composite

$$C \xrightarrow{\alpha^{-1}} M \xrightarrow{\chi_r} M \otimes D \xrightarrow{(\varepsilon_C \alpha) \otimes D} I \otimes D \cong D$$

where χ_r is the right coaction of M. It is routine to show that f is a comonoid morphism and α is an isomorphism of comodules $M \cong f_*$. $\square$

In [13, Observation 8.4] the Eilenberg–Moore construction was described for a comonad in **Comod**$(\mathscr{V})$, and hence for monads with right adjoint. Suppose $G : C \to C$ is a comonad in the bicategory of comodules, with counit $\epsilon : G \to C$ and comultiplication $\delta : G \to G \square_c G$. Then G has a structure of comonoid with counit the composite of ϵ and $\varepsilon : C \to I$, and comultiplication the composite of δ and the equalizer $G\square_C G \rightarrowtail G \otimes G$. With this structure, ϵ is a comonoid morphisms and ϵ_* is an Eilenberg–Moore construction for G.

Now assume that $\mathscr{V}$ is braided. Then the tensor product of $\mathscr{V}$ induces structures of a monoidal bicategory on $\mathbf{Comon}(\mathscr{V})$ and $\mathbf{Comod}(\mathscr{V})$. Moreover, the pseudofunctor $(-)_*$ is strong monoidal. Normal pseudomonoids, that is, pseudomonoids whose unit constraints λ, ρ are identities, in $\mathbf{Comon}(\mathbf{Vect})$ are *coquasibialgebras*. Coquasibialgebras and coquasi Hopf algebras can be found for example in [16,2,19].

The bicategory $\mathbf{Comod}(\mathscr{V})$ is not just monoidal but it is also left and right autonomous. The right bidual of a comonoid C is the opposite comonoid C°. The coevaluation $\mathrm{n} : I \to C^\circ \otimes C$ and evaluation $\mathrm{e} : C \otimes C^\circ \to I$ comodules are the object C with coactions depicted below, where c denotes the braiding of $\mathscr{V}$.

$$\mathrm{n} : C \xrightarrow{\Delta^3} C \otimes C \otimes C \xrightarrow{c\otimes 1} C \otimes C \otimes C \qquad \mathrm{e} : C \xrightarrow{\Delta^2} C \otimes C \otimes C \xrightarrow{1\otimes c} C \otimes C \otimes C$$

Example 8.2. As shown in [4], coquasi-Hopf algebras are exactly the left autonomous normal pseudomonoids in $\mathbf{Comod}(\mathbf{Vect})$ whose unit, multiplication and dualization are representable by coalgebra morphisms.

Recall that for a monoidal monad $G : C \to C$ on a pseudomonoid (in an arbitrary monoidal bicategory), its Eilenberg–Moore object C^G has a canonical structure of a pseudomonoid such that the universal $C^G \to C$ is strong monoidal.

Lemma 8.3. *Let (C, j, p) be a pseudomonoid in $\mathbf{Comon}(\mathscr{V})$ and (G, ϵ, δ) a monoidal comonad in $\mathbf{Comod}(\mathscr{V})$ on the pseudomonoid (C, j_*, p_*). Then, the pseudomonoid structure on the Eilenberg–Moore object of G described above comes from a pseudomonoid structure in $\mathbf{Comon}(\mathscr{V})$.*

Proof. The Eilenberg–Moore construction of G is given by $\epsilon_* : C^G \to C$, where C^G has underlying object $G \in \mathscr{V}$ and $\epsilon : G \to C$ is the counit of the comonad. The pseudomonoid structure on C^G is defined by the existence of canonical isomorphisms

$$\begin{tikzcd} C^G \otimes C^C \arrow[r, "P"] \arrow[d, "\epsilon_*\otimes\epsilon_*"'] \arrow[dr, phantom, "\cong"] & C^G \arrow[d, "\epsilon_*"] \\ C \otimes C \arrow[r, "p_*"'] & C \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "J"] \arrow[dr, "j^*"'] & C^G \arrow[d, "\epsilon_*"] \arrow[dl, phantom, "\cong"] \\ & C \end{tikzcd}$$

To show that P and J are induced by comonoid morphisms, we apply Lemma 8.1. Recall that the counit of the comonoid C^G is the composition of $\epsilon : G \to C$ with the counit of C, $\varepsilon : C \to I$. Then, $(\varepsilon\epsilon)_* P \cong (\varepsilon)_* p_*(\epsilon_* \otimes \epsilon_*) \cong (\varepsilon\epsilon \otimes \varepsilon\epsilon)_*$, and it follows that $P \cong q_*$ for a comonoid morphism q. Similarly, $J \cong k_*$ for a comonoid morphism k. We transport the pseudomonoid structure of (C^G, J, P) to (C^G, k_*, q_*); any pseudomonoid of the latter form comes from a pseudomonoid in $\mathbf{Comon}(\mathscr{V})$, because the pseudofunctor $(-)_*$ from $\mathbf{Comon}(\mathscr{V})$ to $\mathbf{Comod}(\mathscr{V})$ is locally fully faithful. $\square$

8.2. Centre and Drinfel'd double

We now consider the results of Section 6 on the lax centre in the context of comodules. We suppose the underlying monoidal category $\mathscr{V}$ is symmetric, and thus $\mathbf{Comon}(\mathscr{V})$ is a symmetric monoidal $\mathbf{Cat}$-enriched category. Via the monoidal pseudofunctor $(-)_*$ we obtain comodules $c_{M,N} : M \otimes N \to N \otimes M$ making the usual diagrams commute up to canonical isomorphisms in $\mathbf{Comod}(\mathscr{V})$.

Proposition 8.4. *Any left autonomous pseudomonoid in $\mathbf{Comod}(\mathscr{V})$ whose underlying object in $\mathscr{V}$, has a dual has a lax centre. If the pseudomonoid is also right autonomous then the lax centre equals the centre. Furthermore, if the pseudomonoid is induced by a pseudomonoid in $\mathbf{Comon}(\mathscr{V})$, so is its lax centre.*

Proof. At the end of [13] it is noted that any left autonomous pseudomonoid C in $\mathbf{Comod}(\mathscr{V})$ is a *map* pseudomonoid. By Theorem 6.9 we have to show that the monad $\tilde{s} : A \to A$ has an Eilenberg–Moore construction, and for that it is enough to show that it has a right adjoint, since $\mathbf{Comod}(\mathscr{V})$ has Eilenberg–Moore objects for comonads. By Theorem 6.9, we have $\tilde{s} \cong p(p \otimes C)(C \otimes c_{C,C})(p^* \otimes C)(j \otimes C)$ and therefore $\tilde{s}$ has a right adjoint if $p^* j : I \to C \otimes C$ has one; but as C is left autonomous, this 1-cell is isomorphic to $(d \otimes C)\mathrm{n}$ which is a composition of maps: d by [4, Prop. 1.2] and n by [4, Prop. 5.1].

Finally, $\tilde{s}^*$ is a monoidal comonad because $\tilde{s}$ is an opmonoidal monad (see Theorem 6.9). Then, Lemma 8.3 implies that if C comes from a pseudomonoid in $\mathbf{Comon}(\mathscr{V})$, then so does $C^{\tilde{s}^*} = C^{\tilde{s}}$. $\square$

Example 8.5. The proposition above implies that any finite-dimensional coquasi-Hopf algebra H has a lax centre in $\mathbf{Comod}(\mathbf{Vect})$. Moreover, the antipode of a finite-dimensional coquasi-Hopf algebra is always invertible by [1,20]. This means that the dualization of the induced map pseudomonoid is an equivalence, and hence we have a left and right autonomous pseudomonoid (see [4, Prop. 1.5]). It follows that H has a centre and it coincides with the lax centre. Moreover, the lax centre of H can be taken to be a coquasibialgebra.

Observation 8.6. In the proposition above, suppose that the full subcategory $\mathscr{V}_f$ of objects with a dual in $\mathscr{V}$ is closed under equalizers of reflexive pairs. Then the lax centre $Z_\ell(C) \to C$ lies in $\mathbf{Comod}(\mathscr{V}_f)$, and it is a lax centre in it.

To prove this observe that $t : C^\circ \otimes C \to C^\circ \otimes C$ and its Eilenberg–Moore construction $C \to C^\circ \otimes C$ lie in $\mathbf{Comod}(\mathscr{V}_f)$, and the monad s and the distributive law between t and s do so too; see the description of Eilenberg-More constructions for comonads in Section 8.1 or [13, Observation 8.4]. It follows that the induced monad $\tilde{s}$ on C lies in $\mathbf{Comod}(\mathscr{V}_f)$, and it has right adjoint in this bicategory, as shown in the proof above, and it is necessarily the same as in $\mathbf{Comod}(\mathscr{V})$. It follows from the description of Eilenberg–Moore objects mentioned above that $\tilde{s}^*$ has an Eilenberg–Moore construction in $\mathbf{Comod}(\mathscr{V}_f)$ and coincides with the respective construction in $\mathbf{Comod}(\mathscr{V})$. Moreover, this construction is given by $\epsilon_* : C^{\tilde{s}^*} \to C$, where ϵ is the comonoid morphism induced by the counit of the comonad $\tilde{s}^*$. Therefore, the lax centre of C in $\mathbf{Comod}(\mathscr{V}_f)$ is the lax centre of C in $\mathbf{Comod}(\mathscr{V})$.

The *Drinfel'd double* or *quantum double* of a finite-dimensional Hopf algebra is a finite-dimensional braided (also called quasitriangular) Hopf algebra $D(H)$ with underlying vector space $H^* \otimes H$ (one can also take $H \otimes H$) and suitably defined structure. It is a classical result that the category of left $D(H)$-modules is monoidally equivalent to the category of (two-sided) H-Hopf modules and to the centre of the category of H-modules. The Drinfel'd double of a finite-dimensional *quasi-Hopf algebra* was defined in [15] using a reconstruction theorem, and explicit constructions were given in [10,18]. This last paper shows that the category of $D(H)$-modules is monoidally equivalent to the centre of the category of H-modules, via a generalisation of the Yetter–Drinfel'd modules. The quantum double of a coquasi-Hopf algebra was described in [2]. Alternatively, it can be described by dualizing the explicit constructions for the quasi-Hopf case. Then the Drinfel'd or quantum double $D(H)$ of a finite-dimensional coquasi-Hopf H algebra is finite-dimensional and has the property that the category of $D(H)$-comodules $\mathrm{Comod}(D(H))$ is monoidally equivalent to the centre of $\mathrm{Comod}(H)$, and the equivalence commutes with the forgetful functors.

Given a finite-dimensional coquasi-Hopf algebra H, we would like to study the relationship between the centre $Z(H)$ in $\mathbf{Comod}(\mathbf{Vect})$ and the Drinfel'd double $D(H)$. To this aim we will need some of the machinery of *Tannakian reconstruction*, of which we give the most basic aspects following [17].

Let $\mathscr{V}$ be a monoidal category and $\mathscr{V}_f$ the full sub-monoidal category with objects with left duals. We denote by $\mathscr{V}_f$-Act the 2-category of pseudoalgebras for the pseudomonad $(\mathscr{V}_f \times -)$ on **Cat**. Objects of this 2-category are pseudoactions of $\mathscr{V}_f$ and 1-cells are pseudomorphisms of pseudoactions. Observe that $\mathscr{V}_f$ has a canonical $\mathscr{V}_f$-pseudoaction given by the tensor product. We form the 2-category $\mathscr{V}_f$-Alg/$\mathscr{V}_f$ with objects 1-cells $\sigma : \mathscr{A} \to \mathscr{V}_f$ in $\mathscr{V}_f$-Act. The 1-cells are pairs $(F, \phi) : \sigma \to \sigma'$ where $F : \mathscr{A} \to \mathscr{A}'$ is a 1-cell in $\mathscr{V}_f$-Act and $\phi : \sigma' F \cong \sigma$ is a 2-cell in $\mathscr{V}_f$-Act. 2-cells $(F, \phi) \Rightarrow (F', \phi')$ are just 2-cells $F \Rightarrow F'$ in $\mathscr{V}_f$-Act. There is a 2-functor $\mathrm{Comod}_f : \mathbf{Comon}(\mathscr{V}) \to \mathscr{V}_f\text{-Act}/\mathscr{V}_f$ sending a comonoid C to the forgetful functor $\omega_C : \mathrm{Comod}_f(C) \to \mathscr{V}_f$; here $\mathrm{Comod}_f(C)$ is the category of right coactions of C with underlying object in $\mathscr{V}_f$. This category has a canonical $\mathscr{V}_f$-pseudoaction such that ω is an object of $\mathscr{V}_f$-Act/$\mathscr{V}_f$. The definition of Comod_f on 1-cells and 2-cells should be more or less obvious; see [17].

Under certain hypothesis on $\mathscr{V}$, the 2-functor Comod_f is bi-fully faithful. Here is the case we will need: the 2-functor

$$\mathrm{Comod}_f : \mathbf{Comon}(\mathbf{Vect}) \to \mathbf{Vect}_f\text{-Act}/\mathbf{Vect}_f$$

is bi-fully faithful. Moreover, Comod_f is a weak monoidal 2-functor, so that it induces a 2-functor $\mathbf{Mon}(\mathrm{Comod}_f)$. This 2-functor fits in a bi-pullback diagram of 2-functors

$$\begin{tikzcd}
\mathbf{Mon}(\mathbf{Comon}(\mathbf{Vect})) \arrow[r, "\mathbf{Mon}(\mathrm{Comod}_f)"] \arrow[d] & \mathbf{Mon}(\mathbf{Vect}_f\text{-Act}/\mathbf{Vect}_f) \arrow[d] \\
\mathbf{Comon}(\mathbf{Vect}) \arrow[r, "\mathrm{Comod}_f"] & \mathbf{Vect}_f\text{-Act}/\mathbf{Vect}_f
\end{tikzcd}$$

We refer the reader to [17] for a proof of this result.

Recall that ZH can be taken as the Eilenberg–Moore object $H^{\tilde{s}}$.

Theorem 8.7. *For any finite-dimensional coquasi-Hopf algebra H, $H^{\tilde{s}^*}$ and $D(H)$ are equivalent coquasibialgebras. Moreover, they are isomorphic as coalgebras.*

Proof. By Observation 8.6, $H^{\tilde{s}^*}$ is a centre for the pseudomonoid H in $\mathbf{Comod}(\mathbf{Vect}_f)$. Hence we have a monoidal equivalence in $\mathbf{Vect}_f$-Act/$\mathbf{Vect}_f$ from the forgetful functor $\mathrm{Comod}_f(H^{\tilde{s}^*}) \to \mathbf{Vect}_f$ to the forgetful functor $Z(\mathrm{Comod}_f(H)) \to \mathbf{Vect}_f$. On the other hand, there is a monoidal equivalence from the latter to $\mathrm{Comod}_f(D(H)) \to \mathbf{Vect}_f$. In this way we get a monoidal equivalence from $\mathrm{Comod}_f(H^{\tilde{s}^*})$ to $\mathrm{Comod}_f(D(H))$ in $\mathbf{Vect}_f$-Act/$\mathbf{Vect}_f$. By the result mentioned above this theorem, we have a monoidal equivalence $f : H^{\tilde{s}^*} \to D(H)$ in $\mathbf{Comon}(\mathbf{Vect})$. That is, both coquasibialgebras are equivalent. As every equivalence in $\mathbf{Comon}(\mathscr{V})$ has an invertible underlying arrow in $\mathscr{V}$, we deduce that f is an isomorphism of coalgebras. □

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